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Brief paper

Robustness of exponential stability of a class of stochastic functional differential equations with infinite delay ☆

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ABSTRACT

We regard the stochastic functional differential equation with infinite delay $\mathrm{d}x(t) = f(x_t)\mathrm{d}t + g(x_t)\mathrm{d}w(t)$ as the result of the effects of stochastic perturbation to the deterministic functional differential equation $\dot{x}(t) = f(x_t)$, where $x_t = x_t(\theta) \in C((-\infty, 0]; \mathbb{R}^n)$ is defined by $x_t(\theta) = x(t+\theta), \theta \in (-\infty, 0]$. We assume that the deterministic system with infinite delay is exponentially stable. In this paper, we shall characterize how much the stochastic perturbation can bear such that the corresponding stochastic functional differential system still remains exponentially stable.

1. Introduction

It is well known that noise can stabilize a given unstable system (Appleby & Mao, 2005; Appleby, Mao, & Rodkina, 2008; Boulanger, 2000; Caraballo, Garrido-Atienza, & Real, 2003; Hu & Mao, 2008; Khasminskii, 1981; Mao, 1997, 2007; Mao, Yin, & Yuan, 2007; Scheutzow, 1993), as well as destabilize a given stable system (Mao, 1997; Mao et al., 2007; Scheutzow, 1993). Stochastic systems therefore attract increasing attention in many branches of science and industry. An area of particular interest has been the automatic control of stochastic systems, with consequent emphasis being placed on the analysis of stabilization and destabilization.

Consider that a given two-dimensional exponentially stable system

$$\dot{y}(t) = -y(t) \quad \text{on } t \geq 0, \qquad y(0) \in \mathbb{R}^2 \tag{1}$$

is perturbed by noise and the stochastically perturbed system is described by the Itô stochastic equation

$$\mathrm{d}x(t) = -x(t)\mathrm{d}t + Gx(t)\mathrm{d}w(t), \quad x(0) = y(0) \in \mathbb{R}^2 \tag{2}$$

on $t \geq 0$, where $w(t)$ is a scalar Brownian motion and $G \in \mathbb{R}^{2\times 2}$. It has been shown that Eq. (2) has the explicit solution

$$x(t) = \exp\left[-\left(I + \frac{1}{2}G^2\right)t + Gw(t)\right]x(0), \tag{3}$$

where I represents the identity matrix. When

$$G = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, \tag{4}$$

we have $x(t) = \exp[It + Gw(t)]x(0)$, which implies that

$$\lim_{t\to\infty} \frac{\log|x(t)|}{t} = 1, \quad a.s.$$

Clearly, the stochastically perturbed system (2) is almost surely exponentially unstable when G is defined by (4). By (3),

$$\begin{aligned} x(t) &\leq \exp[-It + Gw(t)]\exp\left(\frac{|G|^2 t}{2}\right)|x(0)|\vec{1} \\ &\leq \exp\left[-\left(1 - \frac{|G|^2}{2}\right)It + Gw(t)\right]|x(0)|\vec{1}, \end{aligned}$$

where $|G|$ denotes the trace norm of G and $\vec{1} = (1, 1)^{\mathrm{T}}$. Hence, if $|G| < \sqrt{2}$,

$$\limsup_{t\to\infty} \frac{\log|x(t)|}{t} < 0, \quad a.s.$$

which shows that the stochastically perturbed system (2) is still stable. Note that G represents the intensity of stochastic perturbation imposed on the system (1). This example shows that the stable systems may bear some weak stochastic noise, which implies robustness.

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Stability plays an important role in many system control problems. However, any systems are often subject to noise perturbation. Like the stochastic system (2), noise may destabilize a stable system. It becomes therefore interesting to control noise intensity to make it not destroy the stability of the corresponding deterministic system. Robust stability of stochastic systems recently received more and more attention (Lu, Tsai, Jong, & Su, 2003; Mao, 1992, 1996; Mao, Koroleva, & Rodkina, 1998; Yuan & Mao, 2004).

In this paper, we shall consider stochastic functional differential equations with infinite delay and discuss the robust exponential stability of these systems. Infinite-delay equations now receive the increasing attention since they include many important systems, for example, the pantograph equation in applied mathematics and engineering (see Iserles (1993, 1997) and the references therein) and infinite-delay Volterra equations in mathematical biology and neural networks (for example, Teng (2002) and Zhang, Suda, and Iwasa (2004)). Assume that we are given a general n-dimensional functional differential equation with infinite delay

$$\dot{x}(t) = f(x_t), \tag{5}$$

where $f = (f_1, \dots, f_n)^{\mathrm{T}} : C((-\infty, 0]; \mathbb{R}^n) \to \mathbb{R}^n$ and $x_t = x_t(\theta) \in C((-\infty, 0]; \mathbb{R}^n)$ is defined by $x_t(\theta) = x(t+\theta), \theta \in (-\infty, 0]$. Assume that (5) is stable and perturbed by noise $g(x_t)\dot{w}(t)$, where $\dot{w}(t)$ is an m-dimensional white noise, that is, $w(t)$ is an m-dimensional Brownian motion, and $g = [g_{ij}] : C((-\infty, 0]; \mathbb{R}^n) \to \mathbb{R}^{n\times m}$ represents noise intensity. Also assume that we are required to control g such that the corresponding stochastic functional differential equation with infinite delay

$$\mathrm{d}x(t) = f(x_t)\mathrm{d}t + g(x_t)\mathrm{d}w(t) \tag{6}$$

is also stable. For the purpose of stability, assume that $f(0) = 0, g(0) = 0$, which means that Eq. (6) admits a trivial solution $x(t) \equiv 0$.

Section 2 gives some necessary notations and definitions. To illustrate our idea clearly, Section 3 examines the general results on existence of the global solution to Eq. (6) and exponential stability of this solution. Applying the result of Section 3, we give the conditions under which we discuss robustness of exponential stability of the system (6) in Section 4. As applications of Section 4, Section 5 examines two examples.

2. Preliminaries

Throughout this paper, unless otherwise specified, we use the following notations. Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^{T}. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\mathrm{trace}(A^{\mathrm{T}}A)}$. Let $\mathbb{R}_+ = [0, \infty)$. Denoted by $C((-\infty, 0]; \mathbb{R}^n)$ the family of continuous functions from $(-\infty, 0]$ to $\mathbb{R}^n$. Similarly, denoted by $BC((-\infty, 0]; \mathbb{R}^n)$ the family of bounded continuous functions from $(-\infty, 0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{\theta\le 0} |\varphi(\theta)| < \infty$, which forms a Banach space. Denoted by $\lambda_{\min}(M)$ the eigenvalue of the matrix M with the smallest real part.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions, that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $w(t)$ be an m-dimensional Brownian motion defined on this probability space. If $x(t)$ is an $\mathbb{R}^n$-valued stochastic process on $t \in \mathbb{R}$, we let $x_t = \{x(t+\theta) : -\infty < \theta \le 0\}$ for $t \ge 0$. In addition, throughout this paper, *const* represents a positive constant, whose precise value or expression is not important. Let $C^2(\mathbb{R}^n; \mathbb{R}_+)$ denote the family of all function $V(x)$ from $\mathbb{R}^n$ to $\mathbb{R}_+$ which are continuous twice differentiable, and define an operator $\mathcal{L}V : C((-\infty, 0]; \mathbb{R}^n) \to \mathbb{R}$ by

$$\mathcal{L}V(\varphi) = V_x(\varphi(0))f(\varphi) + \frac{1}{2}\mathrm{trace}[g^{\mathrm{T}}(\varphi)V_{xx}(\varphi(0))g(\varphi)], \tag{7}$$

where

$$V_x(x) = \left(\frac{\partial V(x)}{\partial x_1}, \dots, \frac{\partial V(x)}{\partial x_n}\right)$$

and

$$V_{xx}(x) = \left(\frac{\partial^2 V(x)}{\partial x_i \partial x_j}\right)_{n\times n}.$$

Let $\mathcal{M}_0$ denote all probability measures μ on $(-\infty, 0]$. For any $\varepsilon > 0$, define

$$\mathcal{M}_\varepsilon := \left\{\mu \in \mathcal{M}_0; \mu_\varepsilon := \int_{-\infty}^0 \mathrm{e}^{-\varepsilon\theta}\mathrm{d}\mu(\theta) < \infty\right\}. \tag{8}$$

Clearly, there exist many such probability measures and here we give two examples.

(i) For any $\tau \in \mathbb{R}_+$, let μ be the Dirac measure at $-\tau$ (for definition of the Dirac measure see Kallenberg (1997, P11)). Then for any $\mu \in \mathcal{M}_0$ and $\varepsilon \ge 0$,

$$\mu_\varepsilon = \int_{-\infty}^0 \mathrm{e}^{-\varepsilon\theta}\mathrm{d}\mu(\theta) = \mathrm{e}^{\varepsilon\tau} < \infty,$$

which implies $\mu \in \mathcal{M}_\varepsilon$.

(ii) Let $\mathrm{d}\mu(\theta) = \mathrm{e}^{\theta}\mathrm{d}\theta$. Clearly, $\mu(\theta)$ is a probability measure on $(-\infty, 0]$ and for any $\varepsilon \in (0, 1)$,

$$\mu_\varepsilon = \int_{-\infty}^0 \mathrm{e}^{-\varepsilon\theta}\mathrm{e}^{\theta}\mathrm{d}\theta = \frac{1}{1-\varepsilon} < \infty,$$

which also implies $\mu \in \mathcal{M}_\varepsilon$ for any $\varepsilon \in (0, 1)$.

μ_ε has the following nice property. We give it as a lemma.

Lemma 1. *Fix $\varepsilon_0 > 0$. For any $\varepsilon \in (0, \varepsilon_0]$, μ_ε is continuously nondecreasing and satisfies $\mu_{\varepsilon_0} \ge \mu_\varepsilon \ge \mu_0 = 1$ and $\mathcal{M}_{\varepsilon_0} \subseteq \mathcal{M}_\varepsilon \subseteq \mathcal{M}_0$.*

Proof. Fix $\theta \in (-\infty, 0]$. Clearly, $\mathrm{e}^{-\varepsilon\theta}$ is a nondecreasing function on ε. This implies μ_ε is a nondecreasing function on ε and hence $\mu_{\varepsilon_0} \ge \mu_\varepsilon \ge \mu_0 = 1$ and $\mathcal{M}_{\varepsilon_0} \subseteq \mathcal{M}_\varepsilon \subseteq \mathcal{M}_0$ since $\varepsilon \in (0, \varepsilon_0]$. The Levi Theorem (see Kallenberg (1997, Theorem 1.19, P11)) gives continuity. □

Let $L^p((-\infty, 0]; \mathbb{R}^n)$ denote all functions $h : (-\infty, 0] \to \mathbb{R}^n$ such that $\int_{-\infty}^0 |h(s)|^p ds < \infty$. We give the following lemma.

Lemma 2. *Let $\varphi \in BC((-\infty, 0]; \mathbb{R}^n) \cap L^p((-\infty, 0]; \mathbb{R}^n)$ for any $p > 0$. Then for any $q > p$, $\varphi \in L^q((-\infty, 0]; \mathbb{R}^n)$.*

Proof. Clearly, by the definition of the norm in the space $BC((-\infty, 0]; \mathbb{R}^n)$, we have

$$\begin{aligned}\int_{-\infty}^0 |\varphi(\theta)|^q\mathrm{d}\theta &= \int_{-\infty}^0 |\varphi(\theta)|^p|\varphi(\theta)|^{q-p}\mathrm{d}\theta \\ &\le \|\varphi\|^{q-p}\int_{-\infty}^0 |\varphi(\theta)|^p\mathrm{d}\theta < \infty,\end{aligned}$$

which is the desired assertion. □

We also give the boundedness property of polynomial functions as a lemma.

Lemma 3. *For any $h(x) \in C(\mathbb{R}^n; \mathbb{R})$, $\alpha, b > 0$, if $h(x) = o(|x|^\alpha)$ as $|x| \to \infty$, then*

$$\sup_{x\in\mathbb{R}^n}[h(x) - b|x|^\alpha] < \infty. \tag{9}$$

To examine the stability, we will also need the useful convergence theorem of nonnegative semimartingales (Liptser & Shiryaev, 1989; Mao, 1997) which we cite here as a lemma.