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Brief paper

Almost disturbance decoupling and pole placement☆

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ABSTRACT

We revisit here the Almost Disturbance Decoupling Problem (**ADDP**) (Willems, 1981) by state feedback with the objective to solve **ADDP** and simultaneously place the maximal number of poles in the closed-loop solution. Indeed, when **ADDP** is solvable, we show that, whatever be the choice of a particular feedback solution, the obtained closed-loop system always has a set of fixed poles. We characterize these Fixed Poles of **ADDP**. The other (non-fixed) poles can be placed freely, and we characterize the "optimal" solutions (in terms of ad hoc subspaces and feedbacks) which allow us to solve **ADDP** with maximal pole placement. From our contribution, which treats the most general case for studying **ADDP** with maximal, usually partial, pole placement, directly follow the solutions of **ADDP** with complete pole placement (when there are no **ADDP** Fixed Poles) and **ADDP** with internal stability (when all the Fixed Poles of **ADDP** are stable), without requiring the use of stabilizability subspaces, as in Willems (1981). We extend the concept of Self-Bounded Controlled-Invariant Subspaces (Basile & Marro, 1992) to almost ones. An example is proposed that illustrates our contributions.

1. Introduction

The almost disturbance decoupling problem (**ADDP**) was first introduced in Willems (1981); it is an alternative to the traditional disturbance decoupling problem (**DDP**) when this classical **DDP** is not solvable. It also plays a central role in several important problems, such as robust control, decentralized control and non-interacting control (Lin, 1997; Saberi, Stoorvogel, & Sannuti, 2006). It has been intensively studied in Willems (1981), Schumacher (1984), Trentelman (1985) and Weiland and Willems (1989), but, up until now, the question concerning pole placement in conjunction with **ADDP** was still open: only stabilizability or complete pole placement were answered. The questions about the possible existence of fixed poles, about their locations, and about the design of a particular solution which would place at will all the other (non-fixed) poles were open.

The aim of this paper is to show how to get an optimal solution of **ADDP** in the sense of maximal pole placement. In fact, we show, using the so-called geometric approach, that, there exist some finite fixed poles in **ADDP**, i.e. poles which are present in the closed-loop system after applying any state feedback solution of **ADDP**. These finite fixed poles do not depend on the choice of the control law but precisely on the fact that this particular problem is solvable. Furthermore, these **ADDP** finite fixed poles can also be characterized in terms of finite invariant zeros of the open-loop system, as this was done in Malabre, Martinez-Garcia, and Del-Muro-Cuellar (1997) for exact **DDP**, and later considered in Chu (2003) and Ntogramatzidis (2008).

An important consequence of the characterization of the **ADDP** finite fixed poles is that it directly gives an answer to the problem of **ADDP** with internal stability, say **ADDPS**. In the classical approach, as shown above, to solve **ADDPS**, one needs to precise first the stability region and then to handle the associated stabilizability geometric subspaces. With our results, we precisely know the **ADDP** finite fixed poles and we can conclude about the existence of stabilizing solutions just by looking *a posteriori* at their position with respect to the chosen unstable region.

The paper is organized as follows. In Section 2, we introduce some notation and the basic concepts that will be used. In Section 3, we study the pole assignability of almost invariant subspaces. In Section 4, we give the definition of **ADDP** finite fixed poles and in Section 5 their geometric and structural characterizations.

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Section 6 details an example which illustrates our contributions. Most of the detailed proofs are sent to the Appendix.

2. Notation and geometric preliminaries

We consider linear time-invariant disturbed systems $\Sigma(A, B, D, E)$ described by:

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Dd(t) \\ z(t) = Ex(t) \end{cases}$$

where x, u, d, and z are respectively the state, control input, disturbance input, and output to be controlled. These signals belong to the finite dimensional real vector spaces $\mathscr{X}$, $\mathscr{U}$, $\mathscr{Q}$, and $\mathscr{Z}$, respectively.

In this paper, vectors are denoted by lower case letters, matrices/maps by capitals and subspaces by script capitals. If A is a square matrix, then $\sigma(A)$ denotes its spectrum. If $A : \mathscr{X} \longmapsto \mathscr{Y}$ and $\mathscr{V} \subseteq \mathscr{X}$, the restriction of the map A to $\mathscr{V}$ is denoted by $A|\mathscr{V}$. If $\mathscr{V}_1$ and $\mathscr{V}_2$ are A-invariant subspaces and $\mathscr{V}_2 \subseteq \mathscr{V}_1$, the map induced by A in the quotient space $\mathscr{V}_1/\mathscr{V}_2$ is denoted by $A|\mathscr{V}_1/\mathscr{V}_2$. A^{T} denotes the transpose of A. To simplify, we sometimes use $\mathscr{B}$ in place of $Im\ B$, the image of B and $\mathscr{K}$ in place of $Ker\ E$, the kernel of E. $\oplus$ denotes direct sum of subspaces, $\uplus$ denotes union of sets with common elements repeated. $\mathbb{C}^-$ denotes the open left-half complex plane.

Denote $\Sigma_{(A,B)x} := \{x(t) : [0, \infty) \to \mathscr{X};\ x(t)$ is **a.c.** (absolutely continuous), and $\dot{x}(t) - Ax(t) \in Im\ B$ **a.e.** (almost everywhere)$\}$, and $\Sigma_{(A,[B|D])x} := \{x(t) : [0, \infty) \to \mathscr{X};\ x(t)$ is **a.c.**, and $\dot{x}(t) - Ax(t) \in Im\ B + Im\ D$ **a.e.**$\}$.

If $\mathscr{X}$ is a normed vector space, with norm $\|.\|$, and $\mathscr{L}$ a subspace of $\mathscr{X}$, then for any $x \in \mathscr{X}$, its distance to $\mathscr{L}$ is denoted as: $d(x, \mathscr{L}) := \inf_{y\in\mathscr{L}} \|x - y\|$.

For any measurable function, say $W : [0, \infty) \to \mathscr{X}$, we say that $W \in L_p[0, \infty)$ if $\|W\|_{L_p} < +\infty$, where:

$$\|W\|_{L_p} := \begin{cases} \left(\displaystyle\int_0^\infty \|W(t)\|^p\, \mathrm{d}t\right)^{1/p} & \text{for } 1 \le p < \infty \\ \operatorname*{ess\ sup}_{t\ge 0} \|W(t)\| & \text{for } p = \infty. \end{cases}$$

The reachable space of Σ (by the control u) is denoted by $\langle A|\mathscr{B}\rangle := \mathscr{B} + A\mathscr{B} + A^2\mathscr{B} + \cdots + A^{n-1}\mathscr{B}$, where n is the dimension of $\mathscr{X}$.

A subspace $\mathscr{V} \subseteq \mathscr{X}$ is called $(A, \mathscr{B})$ (controlled) invariant if for any $x_0 \in \mathscr{V}$ there exists an input function u such that the corresponding trajectory $x(t) \in \mathscr{V}$ for all $t \ge 0$ with $x(0) = x_0$; or equivalently if there exists $F : \mathscr{X} \to \mathscr{U}$ such that $(A + BF)\mathscr{V} \subseteq \mathscr{V}$ (Trentelman, Stoorvogel, & Hautus, 2001). F is called a friend of $\mathscr{V}$ and we denote $\mathscr{F}(\mathscr{V})$ the set of all such F.

A subspace $\mathscr{R} \subseteq \mathscr{X}$ is called an $(A, \mathscr{B})$ controllability subspace if for any $x_0 \in \mathscr{R}$, and any $x_1 \in \mathscr{R}$, there exists $T > 0$ and an input function u such that the solution of $\Sigma(A, B, D, E)$ with $x(0) = x_0$ satisfies $x(t) \in \mathscr{R}$ for $0 \le t \le T$ and $x(T) = x_1$; or equivalently if there exist $F : \mathscr{X} \to \mathscr{U}$, and $G : \mathscr{Y} \to \mathscr{U}$, with $\mathscr{Y} \subseteq \mathscr{U}$, such that: $\mathscr{R} := \langle A + BF|Im(BG)\rangle$ (Trentelman et al., 2001).

A subspace $\mathscr{V}_a \subseteq \mathscr{X}$ is called an almost $(A, \mathscr{B})$ (controlled) invariant subspace if for any $x_0 \in \mathscr{V}_a$ and for any $\epsilon > 0$ there exists a state trajectory $x(t) \in \Sigma_{(A,B)x}$ with the properties that $x(0) = x_0$ and $d(x(t), \mathscr{V}_a) \le \epsilon$, for any $t \ge 0$; or equivalently if there exists $F_\epsilon : \mathscr{X} \to \mathscr{U}$ such that, for any $x_0 \in \mathscr{V}_a$ and for any $t \ge 0$, $d(\mathrm{e}^{(A+BF_\epsilon)t}x_0, \mathscr{V}_a) \le \epsilon$. F_ϵ is called an ϵ-distance friend of $\mathscr{V}_a$ and we denote $\mathscr{F}_\epsilon(\mathscr{V}_a)$ the set of all such F_ϵ.

A subspace $\mathscr{R}_a \subseteq \mathscr{X}$ is called an almost $(A, \mathscr{B})$ controllability subspace if for any $x_0 \in \mathscr{R}_a$, and any $x_1 \in \mathscr{R}_a$ there exists $T > 0$ such that, for any $\epsilon > 0$ there exists a state trajectory $x(t) \in \Sigma_{(A,B)x}$ with the properties that $x(0) = x_0$, $x(T) = x_1$ and $d(x(t), \mathscr{R}_a) \le \epsilon$, $\forall t \ge 0$.

$\mathscr{V}^*$, the supremal $(A, \mathscr{B})$ (controlled) invariant subspace contained in $\mathscr{K}$, is the limit of the following non-increasing algorithm : $\mathscr{V}^0 = \mathscr{X}$, $\mathscr{V}^{i+1} = \mathscr{K} \cap A^{-1}(\mathscr{B} + \mathscr{V}^i)$. $\mathscr{R}^*$, the supremal $(A, \mathscr{B})$ controllability subspace contained in $\mathscr{K}$, is the limit of the following non-decreasing algorithm : $\mathscr{R}^0 = 0$, $\mathscr{R}^{i+1} = \mathscr{V}^* \cap (A\mathscr{R}^i + \mathscr{B})$. See Basile and Marro (1992) and Wonham (1985).

$\mathscr{R}_a^*$, the supremal almost $(A, \mathscr{B})$ controllability subspace contained in $\mathscr{K}$, is the limit of the following non-decreasing algorithm : $\mathscr{R}_a^0 = 0$, $\mathscr{R}_a^{i+1} = \mathscr{K} \cap (A\mathscr{R}_a^i + \mathscr{B})$. $\mathscr{V}_a^*$, the supremal almost $(A, \mathscr{B})$ invariant subspace contained in $\mathscr{K}$, satisfies: $\mathscr{V}_a^* = \mathscr{V}^* + \mathscr{R}_a^*$. See Willems (1981).

Let us denote by $\mathscr{S}^*$ the limit of the following algorithm: $\mathscr{S}^0 = 0$, $\mathscr{S}^{i+1} = \mathscr{B} + A(\mathscr{K} \cap \mathscr{S}^i)$. $\mathscr{S}^*$ is usually introduced in the context of $(\mathscr{E}, A)$ invariance (dual to $(A, \mathscr{B})$ invariance). In our present context, we prefer to handle it through its almost controllability properties, as established in Willems (1981), namely: $\mathscr{S}^* = A\mathscr{R}_a^* + \mathscr{B}$ and $\mathscr{R}_a^* = \mathscr{K} \cap \mathscr{S}^*$.

Note that all these notions of exact/almost invariance or controllability properties, can easily be defined, similarly, for the "composite" system (let $B_c := [B, D]$), say $\Sigma_c = \Sigma(A, B_c, 0, E)$, i.e. with $\mathscr{U} \oplus \mathscr{Q}$ in place of $\mathscr{U}$. They will be noted, respectively, $\mathscr{V}_c^*$, $\mathscr{R}_c^*$, $\mathscr{R}_{ca}^*$, $\mathscr{S}_c^*$.

In the following, for any given subspace $\mathscr{L} \subseteq \mathscr{X}$, $\mathscr{V}^*(\mathscr{L})$, $\mathscr{R}^*(\mathscr{L})$ and $\mathscr{R}_a^*(\mathscr{L})$ denote, respectively, the supremal $(A, \mathscr{B})$ invariant, $(A, \mathscr{B})$ controllability and almost $(A, \mathscr{B})$ controllability subspace included in $\mathscr{L}$.

Definition 1 (*Willems, 1981*). **ADDP** (the **Almost Disturbance Decoupling Problem in $L_p - L_q$-sense**) is solvable if : $\forall\epsilon > 0$, there exists a state feedback matrix F_ϵ, such that, in the closed-loop system with $x(0) = 0$, $\|z(t)\|_{L_q} \le \epsilon \|d(t)\|_{L_p}$, for all L_p measurable disturbance input $d(t)$ and for all $1 \le p \le q \le \infty$.

The following Lemma provides a practical method to check whether a state feedback sequence $\{F_\epsilon\}$ can really solve **ADDP**. It will be used in the treatment of our illustrative example.

Lemma 2. (Trentelman, 1983) *Suppose there exists a sequence $\{F_\epsilon\}$ such that $\left\|E\mathrm{e}^{(A+BF_\epsilon)t}D\right\|_{L_p} \xrightarrow{\epsilon\to 0} 0$ for $p = 1$ and $p = \infty$, then* **ADDP** *is solvable.*

From Willems (1981), **ADDP** is solvable if and only if:

$$Im\ D \subseteq \mathscr{V}_a^*\ . \quad (1)$$

Definition 3. (Basile & Marro, 1992) A controlled-invariant subspace included in $\mathscr{K}$, say $\mathscr{V}$, is called self-bounded if $\mathscr{B} \cap \mathscr{V} = \mathscr{B} \cap \mathscr{V}^*$, or equivalently if $\mathscr{R}^*$ is included in $\mathscr{V}$. We denote the set of all the self-bounded controlled-invariant subspaces included in $\mathscr{K}$ by $SBCI(A, B, \mathscr{K})$.

Some particular system structures play a key role in the solution of control problems, among them are the invariant zeros. The finite invariant zeros of $\Sigma(A, B, 0, E)$, i.e. from u to z, are equal to the eigenvalues of $(A + BF)$ in $\mathscr{V}^*/\mathscr{R}^*$, which are the same for any $F \in \mathscr{F}(\mathscr{V}^*)$: $Z(A, B, E) := \sigma(A + BF|(\mathscr{V}^*/\mathscr{R}^*))$.

3. Pole assignability for almost invariant subspaces

It is well known from Schumacher (1980) that complete freedom in placing the closed-loop poles is not usually possible when the state feedback F is restricted to be a friend of a given $(A, \mathscr{B})$ invariant subspace $\mathscr{V}$, namely, there exists some fixed part in $\sigma(A+BF)$ when F is restricted to $\mathscr{F}(\mathscr{V})$. In fact, a similar restriction also exists when using ϵ-distance friends, say $F_\epsilon \in \mathscr{F}_\epsilon(\mathscr{V}_a)$ of a given almost $(A, \mathscr{B})$ invariant subspace $\mathscr{V}_a$.

Lemma 4. *Let $\mathscr{V}_a$ be an almost invariant subspace. For any given symmetric spectra of ad hoc lengths, say Λ_1 and $\Lambda_2 \subset \mathbb{C}^-$, for any*