Automatica 45 (2009) 283–290

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

A new expression for the H^2 performance limit based on state-space representation☆

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ARTICLE INFO

Article history:
Received 4 October 2007
Received in revised form
26 May 2008
Accepted 21 July 2008
Available online 2 December 2008

Keywords:
Performance limits
Infinite-dimensional systems
Non-minimum phase systems
State-space representations

ABSTRACT

This paper aims to clarify the performance limit via state-space representation, in order to handle multiple-input-multiple-output systems in a transparent way. To this end, by applying some results on infinite-dimensional control theory, a closed formula for the H^2 performance limit is derived for systems with a special structure described in terms of a rational transfer matrix and a scalar inner function. This formulation is capable of dealing with various control problems, including H^2 control of a class of infinite-dimensional systems. The resulting formula, given as a functional of the inner function, helps us to understand how achievable H^2 performance deteriorates due to non-minimum phase properties or unstable modes of plants.

1. Introduction

The analysis of performance limitation, one of the most classical topics in control theory, has been paid renewed attention in recent years; see Bakhtiar and Hara (2006), Chen, Hara, and Chen (2003) and Seron, Braslavsky, and Goodwin (1997) and references therein. This research aims to clarify the relationship between the best achievable control performance and plant parameters such as unstable poles/zeros, I/O delay length or weighting functions. These results help us to characterize *easily controllable* plants, or even to *design* plants if these parameters can be chosen by the designer, within some constraints. Thus, for this purpose, it is desirable that the obtained formula should satisfy the following requirements:

- plant parameters remain explicit,
- simple enough to allow intuitive interpretation, and
- capable of dealing with as large a class of plants (or control problems) as possible.

It is, however, not easy to satisfy these conflicting requirements simultaneously.

In this paper, we focus on H^2 control problems. Concerning the third requirement, the standard H^2 control problem can cover a large class of these problems. For this problem, a general solution via matrix Riccati equations is well-known; see Theorem 3 in Section 2. However, it is obviously difficult to understand the relationship between the achievable performance and plant parameters (in this case, system matrices) because the analytical solution to the corresponding matrix Riccati equations is not obtainable. Hence, most of the existing closed-form expressions of the best achievable performance are based on *transfer function representation*, which is adequate for handling plant parameters directly. These results were extended to MIMO cases, e.g., Bakhtiar and Hara (2006), Qiu and Davison (1993) and Su, Qiu, and Chen (2007). However, from a modern control theoretic point of view, this is not the only possible line of research to pursue.

The goal of this paper is to derive a closed formula based on *state-space representation*. To this end, we confine ourselves to generalized plants with a special structure represented by a transfer matrix and an inner function. This formulation covers a large class of control problems, and enables us to derive a new analytical expression for the best achievable H^2 control performance via infinite-dimensional control theory. While the obtained formula uses the solutions of a couple of Riccati equations, the plant non-minimum phase factor or unstable modes, both of which are represented by inner function, remains explicit. This result clarifies how these properties degrade the performance limit in the framework of the standard H^2 control problem.

This paper is organized as follows: in the next section, some preliminary results are given. In Section 3, Theorem 4, the main result of this paper, is derived. Section 3.3 provides two other equivalent

☆ This paper was partially presented at European Control Conference 2007, Kos, July, 2007. This paper was recommended for publication in revised form by Associate Editor Tongwen Chen under the direction of Editor Ian R. Petersen.

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0005-1098/$ – see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2008.07.008

expression, which are suitable for the interpretation and the computation of Theorem 4. In Section 5, we investigate a dual problem which enables us to clarify the effect of unstable modes.

2. Preliminaries

2.1. Definition and notation

For a complex function matrix f, its para-Hermitian conjugate is denoted by $f^{\sim}(s) := \overline{f(-\bar{s})}^{\mathsf{T}}$ where X^{T} is the transpose of X. As usual, H^p (H^p_-) is the Hardy p-space on the open right (left) half complex plane, respectively. In particular, H^2 and H^2_- are Hilbert spaces with the following inner product:

$$\langle f, g\rangle := \operatorname{trace}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} g^{\sim}(j\omega)f(j\omega)\mathrm{d}\omega\right). \tag{1}$$

Let L^2 be the Lebesgue space of square integrable functions on the imaginary axis. Function space L^2 is also a Hilbert space with the inner product defined by (1) such that $L^2 = H^2 \oplus H^2_-$. The norm of Hilbert space H is denoted by $\|f\|_H := \sqrt{\langle f, f\rangle_H}$. For simplicity, H^2-norm is represented by $\|\cdot\|_2$. For state-space realization of rational transfer matrix,

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := D + C(sI - A)^{-1}B.$$

The size of matrices is omitted as it is clear from the context.

A scalar function $m \in H^\infty$ is said to be *inner* if $|m(s)| = 1$ a.e. on the imaginary axis. Since any inner function satisfies

$$mm^{\sim} = 1 \tag{2}$$

a.e. on the imaginary axis, the domain of m is analytically extended to $\mathbb{C}_-$ by using this equality, i.e., $m = 1/m^{\sim}$. Therefore, zeros (respectively, poles) of m are poles (respectively, zeros) of $m^{\sim}$. For any inner m, mH^2 is right shift invariant subspace in H^2. Let $H(m)$ be the orthogonal complement of mH^2 on H^2. Function space $H(m)$ is left shift invariant and satisfies

$$H(m) = \{x \in H^2 : m^{\sim}x \in H^2_-\}. \tag{3}$$

From (3) and $m^{\sim} = 1/m$, every singularity of any element of $H(m)$ is a pole of m. See also Kashima and Yamamoto (2005) and references therein for other properties of $H(m)$.

For a scalar complex function f, let $\mathcal{M}_f$ be the set of square matrices X such that $m^{\sim}$ is analytic in a neighborhood of every eigenvalue of X. For $X \in \mathcal{M}_f$, matrix function $m^{\sim}(X)$ can be defined as follows (Gantmacher, 1960; Golub & van Loan, 1989):

$$f^{\sim}(X) := \frac{1}{2\pi j}\int_{\Delta} f^{\sim}(s)(sI - X)^{-1}\mathrm{d}s, \tag{4}$$

where the closed contour Δ encircles all eigenvalues of X counterclockwise and $f^{\sim}$ is analytic inside Δ.

The *generalized m-truncation* $\pi^m[\cdot]$ plays a crucial role in this paper (Kashima & Yamamoto, 2008a; Ohta, 2005):

Definition 1. Let m be an inner function and $W(s) = C(sI - A)^{-1}B$ with $A \in \mathcal{M}_m$. Define

$$W^{(m)} := \left[\begin{array}{c|c} A & m^{\sim}(A)B \\ \hline C & 0 \end{array}\right] \tag{5}$$

$$\pi^m[W] := W - mW^{(m)} \in H(m). \tag{6}$$

This definition does not depend on the choice of the realization of W. It should be noted that $\pi^m[W]$ is stable even if W is unstable.

2.2. Standard H^2 control problem

For transfer matrices P and C of appropriate dimensions, we say that C internally stabilizes P if all nine transfer matrices from w, u_1, u_2 to z, v_1, v_2 in Fig. 1 belong to H^∞. The (lower) linear fractional transform is denoted by $\mathcal{F}_l(\cdot,\cdot)$, that is, $\mathcal{F}_l(P, C)$ is the transfer matrix from w to z in Fig. 1.

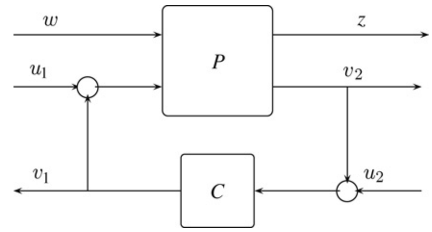

Fig. 1. Definition of internal stability.

For the generalized plant

$$\check{P} := \left[\begin{array}{c:c} \check{P}_{11} & \check{P}_{12} \\ \hdashline \check{P}_{21} & \check{P}_{22} \end{array}\right] := \left[\begin{array}{c|c:c} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ \hdashline C_2 & D_{21} & D_{22} \end{array}\right] \tag{7}$$

satisfying Assumption 2, the standard H^2 control problem can be solved as follows (Doyle, Glover, Khargonekar, and Francis (1989); Zhou, Doyle, and Glover (1995)):

Assumption 2. (1) (A, B_2) is stabilizable and (A, C_2) is detectable.
(2) For any $\omega \in \mathbb{R}$,

$$\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}, \qquad \begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$$

are row- and column-full rank, respectively.
(3) $D_{12}^{\mathsf{T}}D_{12} = I$, $D_{21}D_{21}^{\mathsf{T}} = I$.
(4) $D_{22} = 0$.

Theorem 3. *Suppose that $\check{P}$ in (7) satisfies Assumption 2. Let $X, Y \geq 0$ be stabilizing solutions to the following Riccati equations:*

$$XA + A^TX + C_1^TC_1 - F^TF = 0, \tag{8}$$

$$AY + YA^T + B_1B_1^T - LL^T = 0, \tag{9}$$

where F, L are given by

$$F := -(B_2^TX + D_{12}^TC_1), \tag{10}$$

$$L := -(YC_2^T + B_1D_{21}^T). \tag{11}$$

Then the H^2 performance limit

$$E_n := \min_{\check{C}\in\check{\mathcal{C}}} \left\|\mathcal{F}_l\left(\check{P}, \check{C}\right)\right\|_2 \tag{12}$$

with $\check{\mathcal{C}}$ being the set of internally stabilizing controllers of $\check{P}$, is given by

$$E_n^2 = \operatorname{trace}\left(B_1^TXB_1 + FYF^T\right).$$

For $\gamma > E_n$, all $\check{C} \in \check{\mathcal{C}}$ satisfying $\left\|\mathcal{F}_l\left(\check{P}, \check{C}\right)\right\|_2 < \gamma$ are parameterized by

$$\check{C} = \mathcal{F}_l(M, Q), \tag{13}$$

where

$$M := \left[\begin{array}{c|cc} A + B_2F + LC_2 & -L & B_2 \\ \hline F & 0 & I \\ -C_2 & I & 0 \end{array}\right] \tag{14}$$

and Q is an arbitrary element of

$$\mathcal{Q} := \left\{Q \in H^2 \cap H^\infty : \|Q\|_2^2 < \gamma^2 - E_n^2\right\}. \tag{15}$$

Moreover,

$$\left\|\mathcal{F}_l\left(\check{P}, \mathcal{F}_l(M, Q)\right)\right\|_2^2 = E_n^2 + \|Q\|_2^2. \tag{16}$$