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System theory for numerical analysis☆

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Abstract

Many numerical schemes can be suitably studied from a system theoretic point of view. This paper studies the relationship between the two disciplines, that is, numerical analysis and system theory. We first see that various iterative solution schemes for linear and nonlinear equations can be suitably transformed into the form of a closed-loop feedback system, and show the crucial role of the internal model principle in such a context. This leads to new stability criteria for Newton's method. We then study Runge–Kutta type methods for solving differential equations, and also derive new stability criteria based on recent results on LMI. A numerical example is given to illustrate the advantage of the present theory.

Keywords: Numerical analysis; System theory; Stability of numerical methods; Newton's method; Internal model principle; Lur'e control systems; Absolute stability; Linear matrix inequality

1. Introduction

There exist many iterative numerical schemes for solving linear or nonlinear equations or differential equations, having different characteristics adopted to varied needs. These schemes are mostly represented by difference equations, and hold an important position. Although system theory is clearly a suitable tool for analyzing such dynamical systems, the study of numerical analysis from this viewpoint has not been quite popular. However, recently some authors have started system theoretic approaches toward numerical analysis; see, for example, Gustafsson, Lundh, and Söderlind (1988), Bhaya and Kaszkurewicz (2003, 2004), Kaszkurewicz, Bhaya, and Ramos (1995), Schaerer and Kaszkurewicz (2001); and Wakasa and Yamamoto (2001) and references therein. The crux of these approaches lies in the fact that we can not only describe the behavior of numerical solutions obtained by iterative schemes in such a system theoretic framework, but also study various requirements such as convergence, stability and robustness against external errors from this viewpoint. This fact opens a great opportunity for system theory to provide numerical analysis with valuable new tools and concepts for analyzing or even synthesizing pertinent dynamical systems associated with it.

We start this paper by analyzing the simple linear equation $Ax=b$, to show the generic idea here on one hand: that is, to interpret various iterative solution processes as feedback systems that are to track a constant (step) input b with the 0th order plant A. On the other hand, the important objective here is to show the crucial relevance of the internal model principle in this context. It arises from the objective of tracking b in spite of small computational or data errors arising in the process of computation, and this is how the internal model principle comes into play; see Section 2 for details. We then generalize this idea to nonlinear equations. A stability criteria for Newton's iterative process is derived in a framework of nonlinear control system, especially system of Lur'e type. The internal model principle again plays an important role here. The present analysis

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enables us to relax conditions on the convergence region, and the result is compared with those by conventional analysis.

In Section 2, we turn our attention to numerical integration methods of ordinary differential equations, particularly the analysis of the absolute stability region of Runge–Kutta type methods. The absolute stability at a point in the complex plane means that a corresponding linear test problem is stable. The absolute stability region governs the step size to guarantee accurate numerical solutions. While this is an important problem, it is also known to be difficult to describe relationships between this region and Runge–Kutta coefficients. Only for some special cases, algebraic conditions of the coefficients have been obtained, e.g., Scherer and Türke (1989). We here invoke a new generalized Kalman–Yakubovich–Popov (KYP) lemma derived by Iwasaki and Hara (2005), to obtain a more general characterization of this region in terms of a linear matrix inequality (LMI). This allows us to *design* the coefficients of a Runge–Kutta type method by optimizing the region of absolute stability.

2. Iterative schemes and the internal model principle

The objective of this section is to show the relevance of the internal model principle to various iterative schemes.

2.1. Iterative processes for linear equations

We start with a simple linear equation $Ax = b$. The crux here is to place this into the framework of tracking systems, and show that the internal model principle plays a crucial role.

Let $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, and consider the linear equation

$$Ax = b$$

for $x \in \mathbb{R}^n$. Suppose that A is nonsingular, and we wish to generate a sequence x_k that converges to the solution $A^{-1}b$. Decompose A as

$$A = D + E + F, \tag{1}$$

where D, E and F are diagonal, strictly lower triangular and strictly upper triangular matrices, respectively. Then the Jacobi, Gauss–Seidel (GS), and successive over-relaxation (SOR) methods are given as follows (Quarteroni, Sacco, & Saleri, 2000):

$$\begin{aligned}
\text{Jacobi} : x_{k+1} &= -D^{-1}(E+F)x_k + D^{-1}b,\\
\text{GS} : x_{k+1} &= -(D+E)^{-1}Fx_k + (D+E)^{-1}b,\\
\text{SOR} : x_{k+1} &= (I+\omega D^{-1}E)^{-1}\{(1-\omega)I - \omega D^{-1}F\}x_k\\
&\quad + \omega(D+\omega E)^{-1}b,
\end{aligned}$$

where ω is called a relaxation parameter.

These formulas perhaps give a somewhat ad hoc impression. What is important here is to recognize that they can be brought into the form of a feedback system driven by the error $Ax - b$. This makes it possible to relate such schemes with the internal model principle and provides a unified, and simplified viewpoint.

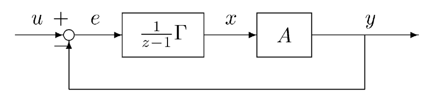

Fig. 1. Block diagram for iterative methods.

Let us first regard A as a 0th order plant to be controlled, and define the error signal $e_k := b - Ax_k$ which is expected to converge to 0. We can rewrite the algorithm of the Jacobi method as

$$\begin{aligned}
x_{k+1} &= -D^{-1}(A-D)x_k + D^{-1}b\\
&= x_k + D^{-1}(b - Ax_k)\\
&= x_k + D^{-1}e_k.
\end{aligned}$$

Similarly, the other two methods can also be brought into a feedback form

$$x_{k+1} = x_k + \Gamma e_k$$

with Γ given by $(D+E)^{-1}$ for GS and $\omega(D+\omega E)^{-1}$ for SOR. It is readily obvious that these methods take the common form of Fig. 1, with step input signal $u \equiv b$. We now attempt to see a more intrinsic reason for this.

Let us start by observing that any reasonable iterative solver should satisfy the following properties:

- For arbitrary b, the method should work, i.e., the output $y_k = Ax_k$ should track arbitrary constant b in order that x_k converge to the exact solution.
- This tracking property is robust, in presence of some data errors, the method should still converge.

The celebrated internal model principle (Francis & Wonham, 1975) asserts that this robust tracking property is satisfied if and only if the following two conditions hold:

(i) the feedback system is internally stable, i.e., the transfer matrix from u to y is stable and no unstable pole-zero cancellation exists; and
(ii) the loop transfer matrix from e to y contains the internal model of exogenous signals, which is the step signal generator $1/(z-1)$ in this case.

If we further assume uncertainty in the plant, property (ii) *forces* the controller to contain the integrator $1/(z-1)$, and the simplest of such a construction is that given in Fig. 1. Moreover, this construction rejects an arbitrary constant disturbance added to x in Fig. 1. This tells us that why most, if not all, iterative schemes assume the structure $x_{k+1} = x_k +$ correction term: it is a crucial consequence of the internal model principle.

In Fig. 1, the open-loop transfer matrix clearly contains $1/(z-1)$. Thus x_k converges to $A^{-1}b$ if and only if condition (i) above is satisfied, i.e., all eigenvalues of $I - \Gamma A$ lie in the open unit disc. This is consistent with the conventional results.