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Brief paper

Synchronization of coupled harmonic oscillators in a dynamic proximity network☆

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ABSTRACT

In this paper, we revisit the synchronization problems for coupled harmonic oscillators in a dynamic proximity network. Unlike many existing algorithms for distributed control of complex dynamical networks that require explicit assumptions on the network connectivity, we show that the coupled harmonic oscillators can always be synchronized, without imposing any network connectivity assumption. Moreover, we also investigate the synchronization with a leader and show that all harmonic oscillators can asymptotically attain the position and velocity of the leader, again without any assumption on connectivity of the followers. Numerical simulation illustrates the theoretical results.

1. Introduction

In recent years, there has been significant interest in the study of synchronization from different fields (see, for example, Wang (2002) and Wu (2002) and the references therein). Two main lines of research on the problems of synchronization have emerged from this study. On the one hand, the two pioneering papers on synchronization in coupled systems (Fujisaka & Yamada, 1983) and synchronization in chaotic systems (Pecora & Carroll, 1990) have stimulated a great deal of interest in the study of complete synchronization of coupled nonlinear dynamical systems. On the other hand, there has been much interest in the study of synchronization in dynamical networks with complex topologies in the past few years due to the discovery of the small-world and scale-free properties of many natural and artificial complex networks (Wang & Chen, 2002, 2003).

One of the most important contributions to the problem of synchronization in coupled systems is the Kuramoto model (Kuramoto, 1984), which was established based on the phenomenon of collective synchronization. In the Kuramoto model, the information of oscillators are assumed to be global, *i.e.*, the underlying topology is fully connected. The Kuramoto model was later modified for the scenarios of nearest neighbor interaction (Jadbabaie, Motee, & Barahona, 2004), switching topologies and presence of nonhomogeneous delays (Papachristodoulou & Jadbabaie, 2005). In contrast to the above models where the coupled systems are described by single integrator dynamics, Ren in Ren (2008a) recently investigated coupled second-order linear harmonic oscillator models. The oscillators investigated in Ren (2008a) are modeled as point masses on a real line. Under some rather mild network connectivity assumptions, the positions and velocities of coupled harmonic oscillators can be synchronized in both fixed and switching networks, with or without a leader.

Related to the synchronization of coupled harmonic oscillators are second-order consensus problems (Ren, 2008b; Ren & Atkins, 2007; Xie & Wang, 2007) and flocking problems (Olfati-Saber, 2006; Su, Wang, & Chen, 2009a; Su, Wang, & Lin, 2009b; Su, Wang, & Yang, 2008; Tanner, Jadbabaie, & Pappas, 2007) in multi-agent systems. In order to achieve second-order consensus in a multi-agent system, the underlying topology must contain a directed spanning tree in fixed networks, or must have a directed spanning tree at each time instant in switching networks (Ren, 2008b; Ren & Atkins, 2007; Xie & Wang, 2007). In the case of tracking a virtual leader, one of the followers should have the information of the virtual leader in a fixed network (Ren, 2008b). Stimulated by Reynolds' model (Reynolds, 1987), flocking algorithms have been proposed by combining a local artificial potential field with a velocity consensus component (Olfati-Saber, 2006; Su et al., 2009a,b, 2008; Tanner et al., 2007). The convergence condition for

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the flocking algorithms in Olfati-Saber (2006); Tanner et al. (2007) and Su et al. (2008) is that the underlying topology is connected at each time instant. In order to track a virtual leader, the followers should get in touch, directly or indirectly, with the virtual leader from time to time (Su et al., 2009b). The convergence condition for the connectivity-preserving flocking algorithm based only on position measurements is that the initial network is connected (Su et al., 2009a).

Synchronization of coupled harmonic oscillators, second-order consensus and flocking are all characterized by second-order dynamics, distributed control, local interactions and self-organization. A key difference among these problems lies in the intrinsic dynamics of the uncoupled systems. The intrinsic dynamics of the uncoupled systems in the second-order consensus and flocking problems are that of double integrators, while in the synchronization of coupled harmonic oscillators, the dynamics are that of second-order oscillators. A direct consequence of this difference is that the consensus and flocking equilibria for the velocities are zero or nonzero constants, while the synchronization equilibrium for the velocities is time varying.

In the synchronization (Ren, 2008a), second-order consensus (Ren, 2008b; Ren & Atkins, 2007; Xie & Wang, 2007) and flocking (Olfati-Saber, 2006; Su et al., 2009a,b, 2008; Tanner et al., 2007) algorithms, certain network connectivity assumptions play a crucial role in the stability analysis. This is because exchanging sufficient information among agents is necessary for cooperation. However, in practice, such kinds of network connectivity assumptions are usually very difficult to verify and may not hold even if the initial network is well connected. On the other hand, we observe that the intrinsic dynamics of the harmonic oscillators will cause the agents to meet with each other from time to time, even if the initial velocities and positions are different.

Motivated by this observation and inspired by the recent work (Ren, 2008a), we revisit in this paper the coupled second-order linear harmonic oscillator models in a dynamic proximity network. The topology of proximity network depends on the relative distances of the harmonic oscillators. The harmonic oscillators are coupled by their velocity information. We will examine the synchronization of coupled harmonic oscillators in the dynamic proximity network without any connectivity assumption. We will also examine the synchronization of coupled harmonic oscillators with a leader and in the absence of any connectivity assumption on the followers.

The remainder of the paper is organized as follows. Section 2 states the problems to be solved in this paper. Section 3 establishes synchronization results, both without and with a leader. Section 4 presents the simulation results. Finally, Section 5 draws a brief conclusion to the paper.

2. Problem statement

We consider N agents moving in a one-dimensional Euclidean space. The behavior of each agent is described by a harmonic oscillator of the form

$$\begin{aligned} \dot{q}_i &= p_i, \\ \dot{p}_i &= -\omega^2 q_i + u_i, \quad i = 1, 2, \ldots, N, \end{aligned} \tag{1}$$

where $q_i \in \mathbf{R}$ is the position of agent i, $p_i \in \mathbf{R}$ is its velocity vector, $u_i \in \mathbf{R}$ is its control input and ω is the frequency of the oscillator. For notational convenience, we also define

$$q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{bmatrix}, \qquad p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{bmatrix}.$$

The problem of synchronization is to design a control input u_i to cause

$$\lim_{t\to\infty} \|q_i(t) - q_j(t)\| = 0,$$

and

$$\lim_{t\to\infty} \|p_i(t) - p_j(t)\| = 0,$$

for all i and j. In the situation where a leader, labeled as agent $N+1$, is present, the goal is then to design a control input u_i to cause

$$\lim_{t\to\infty} \|q_i(t) - q_\gamma(t)\| = 0,$$

and

$$\lim_{t\to\infty} \|p_i(t) - p_\gamma(t)\| = 0,$$

for all i, where q_γ and p_γ are the position and velocity of the leader, respectively. The dynamic of the leader satisfies

$$\begin{aligned} \dot{q}_\gamma &= p_\gamma, \\ \dot{p}_\gamma &= -\omega^2 q_\gamma. \end{aligned} \tag{2}$$

In Ren (2008a), N coupled harmonic oscillators are connected by dampers, *i.e.*,

$$u_i = -\sum_{i=1}^{N} a_{ij}(t)\left(p_i - p_j\right), \quad i = 1, 2, \ldots, N, \tag{3}$$

where $a_{ij}(t)$ characterizes the interaction between agents i and j at time t. Under certain network connectivity assumptions and the influence of the control input (3), synchronization of the positions and velocities in both fixed and switching networks was established in Ren (2008a).

In this paper, we investigate the system in a dynamic proximity network. Each agent has a limited communication capability which allows it to communicate only with agents within its neighborhood. The neighboring agents of agent i at time t is denoted as:

$$\mathcal{N}_i(t) = \left\{j : \|q_i - q_j\| < r, j = 1, 2, \ldots, N, j \neq i\right\},$$

where $\|\cdot\|$ is the Euclidean norm. In the above definition, we have assumed that all agents have an identical influencing/sensing radius r. During the course of motion, the relative distances between agents may vary with time, so the neighbors of each agent may change. We define the neighboring graph $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t)\}$ to be an undirected graph consisting of a set of vertices $\mathcal{V} = \{1, 2, \ldots, N\}$, whose elements represent agents in the group, and a set of edges $\mathcal{E}(t) = \{(i, j) \in \mathcal{V} \times \mathcal{V} : i \sim j\}$, containing unordered pairs of vertices that represent neighboring relations at time t. Vertices i and j are said to be adjacent at time t if $(i, j) \in \mathcal{E}(t)$. A path of length l between vertices i and j is a sequence of $l + 1$ distinct vertices starting with i and ending with j such that consecutive vertices in the sequence are adjacent.

3. Synchronization of coupled harmonic oscillators

3.1. Synchronization without a leader

Let the control input for agent i be given by

$$u_i = -\sum_{j\in\mathcal{N}_i(t)} a_{ij}(q)\left(p_i - p_j\right), \quad i = 1, 2, \ldots, N, \tag{4}$$

where $A(q) = \left(a_{ij}(q)\right)_{N\times N}$ is the adjacent matrix which is defined in Olfati-Saber (2006) as

$$a_{ij}(q) = \begin{cases} 0, & \text{if } j = i, \\ \rho_h\left(\|q_j - q_i\|_\sigma / \|r\|_\sigma\right), & \text{if } j \neq i, \end{cases}$$