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Brief paper

On the stability of a reduced-order filter based on dominant singular value decomposition of the system dynamics[☆]

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ABSTRACT

The stability of a reduced-order filter (ROF), the gain of which is constructed on the basis of a subspace of dominant singular vectors of the system dynamics, is examined. A definition of s-detectability is introduced. It is found that the observability of all unstable and neutral singular vectors (s-detectability) is a sufficient condition for the existence of a stable filter.

1. Introduction

One of great challenges for future control theory is related to theoretical and practical studies on estimation and control in very high dimensional systems which represent discretized versions of systems of partial differential equations. Such systems are available, for example, in many meteorological and oceanography centers which serve as numerical models for simulating and forecasting the circulation of fluid dynamics. With degree of freedom 10^6–10^7, it is simply impossible to apply the classical control and filtering algorithms (such as the Kalman filter (KF), for example) to these systems.

To deal with filtering problems in very high dimensional systems, a reduced order adaptive filter (ROAF) has been proposed in Hoang, De Mey, Talagrand, and Baraille (1997) and Hoang, Baraille, and Talagrand (2005). In the ROAF, reduction is performed directly on the filter gain. The procedure to determine which parameters of the gain can be adjusted to data and to what extent their values can vary is very important for ensuring stability of the filter. This question has been studied in Hoang, Baraille, and Talagrand (2001) using the eigenvalue decomposition (EVD) of the system dynamics.

A natural approach to filter reduction is to perform direct model reduction on a high-order filter using a model reduction such as balanced truncation (see Fairman, 1977; Moore, 1981; Pernebo & Silverman, 1982), Hankel-norm approximation ((Glover, 1984)), optimal L_2 reduction ((Yan, Xie, & Lam, 1997)) and optimal H_2 reduction ((Xie, Yan, & Soh, 1999)). In Xie et al. (1999), parameterization of a set of filters of fixed order leads to stable filtering error transfer functions for a general class of unstable system dynamics. The question on minimizing the H_2-norm of the filtering error transfer function is also addressed. We note that at first glance the objective of this work is somewhat similar to that studied in Hoang et al. (2001). However, looking at this work in detail reveals that whereas in Xie et al. (1999) the stability of the filter for the reduced state $x_e(t)$ is studied, the objective in Hoang et al. (2001) is to design a stable filter for estimating the full system state $x(t)$, and reduction is performed only on the filter gain. Moreover, in Hoang et al. (2001) the algorithm for gain design is entirely based on the dominant part of the EVD of the system dynamics. On the other hand, Eqs. (10)–(12) (Theorem 1) in Xie et al. (1999) for the ROF depend on the dynamics matrix and on the gain of the full-order KF, which makes the problem of implementation of such filters for very high dimensional dynamical systems unrealizable in practice.

It should be mentioned that by exploiting some dominant directions of error growth (dominant singular vectors) Cohn and Todling (1996) showed from a twin experiment on data assimilation for the two-dimensional, linear shallow-water model that the PSF (Partial

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Singular Value Decomposition Filter) must account for all singular vectors (SVs) with singular values larger or equal to 1, for otherwise the PSF diverges.

This paper focuses on the conditions ensuring stability of the filter based on leading SVs. One of great advantages of the SVD approach is that the leading SVs of a very large sparse matrix can be computed efficiently using the Lanczos algorithm (Golub & van Loan, 1996). Moreover, unlike the eigenvectors (EVs) and eigenvalues which may be complex, the SVs are real and the singular values are positive.

In Section 2, brief descriptions of the PSF and the ROAF are given. In what follows, an ROF with projection subspace constructed on the basis of a partial SVD is denoted an SROF. The notation PSF refers to a Kalman-like version of an SROF (to reserve its original version proposed in Cohn and Todling (1996)). The main results and proofs on stability of the SROF are presented in Section 3. Application of the results in Section 3 to deduce the stability of the PSF will be addressed in Section 4. This study allows us to seek the gains with stabilizing structures for adaptive purposes (Section 5). The conclusion is given in Section 6.

2. PSF and ROAF

2.1. Partial singular value decomposition filter (PSF)

Consider a linear filtering problem for the dynamical system

$$x(t+1) = \Phi x(t) + w(t), \quad (1)$$

$$z(t+1) = Hx(t+1) + v(t+1), \quad t = 0, 1, 2, \ldots. \quad (2)$$

In (1) and (2), $x(t) \in R^n$ is the system state (n is of order 10^6–10^7), $z(t) \in R^p$ is the observation, and the model error $w(t)$ is a zero mean white noise with the covariance $Q(t)$. In (2), H is the known observation operator, and $v(t)$ represents the measurement error which is assumed to be white with zero mean and covariance $R(t)$. The sequences $w(t)$ and $v(t)$ are mutually independent.

The main idea underlying the PSF of Cohn and Todling (1996) is to replace Φ by its partial SVD Φ_1 (see (6) below). Let us introduce the filter

$$\begin{aligned}\hat{x}(t+1) &= \Phi\hat{x}(t) + K(t+1)\zeta(t+1) \\ &= L(t)\hat{x}(t) + K(t+1)z(t+1), \quad (3)\end{aligned}$$

$$L(t) := [I - K(t+1)H]\Phi, \qquad \zeta(t+1) = z(t+1) - H\Phi\hat{x}.$$

In the KF,

$$K(t+1) = M(t+1)H^T[HM(t+1)H^T + R]^{-1}, \quad (4)$$

$$M(t+1) = \Phi P(t)\Phi^T + Q,$$

$$P(t+1) = [I - K(t+1)H]M(t+1).$$

Let $\Phi = UDV^T$ be SVD of Φ. Introduce

$$U = [U_1, U_2], \qquad V = [V_1, V_2], \qquad D = \text{diag}[D_1, D_2] \quad (5)$$

where U_1, V_1 are of dimensions $(n \times \nu)$, U_2, V_2 are $[n \times (n-\nu)]$ matrices, and D_1, D_2 are $(\nu \times \nu)$ and $[(n-\nu) \times (n-\nu)]$ matrices, respectively. The diagonal elements σ_i of D are known as singular values, and the columns of U and V as the *left* and the *right* SVs. We have

$$\Phi = \Phi_1 + \Phi_2, \qquad \Phi_1 := U_1D_1V_1^T, \qquad \Phi_2 := U_2D_2V_2^T \quad (6)$$

where $D_1 = \text{diag}\,[\sigma_1, \ldots, \sigma_\nu]$ is composed from the first ν leading singular values of Φ; U_1 and V_1 are composed from leading left and right SVs. Let ν denote the number of all unstable and neutral singular values of Φ, i.e. $\sigma_k \geq 1$ for all $k = 1, \ldots, \nu$, $\sigma_k < 1$ for $k = \nu+1, \ldots, n$. Roughly speaking, the PEF in Cohn and Todling (1996) is obtained by replacing Φ by Φ_1 in the expressions for $M(t)$, $P(t)$ in (4).

2.2. Reduced-order adaptive filter (ROAF)

In Hoang et al. (1997), the gain K in (3) is postulated to be of the form

$$K = P_rK_e \quad (7)$$

where P_r is an a priori known $n \times n_e$ matrix ($n_e < n$), and K_e is an $(n_e \times p)$ matrix representing a gain in the filter for the reduced state $x_e(t)$ of dimension n_e. Parameterization of K is proposed to be done in $K_e = K_e(\theta)$. The filter for $x(t)$ has the form

$$\hat{x}(t+1) = \Phi\hat{x}(t) + P_rK_e\zeta(t+1). \quad (8)$$

The optimal ROAF is obtained by minimizing the mean prediction error (MPE) of the system output using θ as a control vector. Its application to the problem of estimating the oceanic circulation using sea surface height data in the oceanic model MICOM has been studied in Hoang et al. (2005).

3. Main stability results

3.1. Stability of the SROF

For an n vector x let $\|x\| := \|x\|_2$ denote the l_2 norm of x. The matrix norm is defined as that associated with the l_2 vector norm. Thus for an $(n \times p)$ matrix A, $\|A\| = \sigma_1$, where σ_1 is the largest singular value of A.

Consider the linear stochastic system

$$y(t+1) = A(t)y(t) + \xi(t), \quad t = 1, 2, \ldots, y(0) = y_0 \quad (9)$$

where $\xi(t)$ is a zero mean random sequence of finite variance. Assume that

$$\sqrt{E\|\xi(t)\|^2} \leq \sigma_\xi < \infty \quad (10)$$

where σ_ξ is independent of t. Let $y(t, y_0, \xi)$ be the solution of (9) subject to the initial $y(0) = y_0$ and the input sequence $\xi(t)$.

Definition 3.1. Consider the system (9) and suppose that $y(0)$ is a random vector of finite two first moments. The system (9) is said to be mean square (m.s.) stable if

$$\lim_{t\to\infty} \sqrt{E\|y(t, y_0, \xi)\|^2} < \infty.$$

Subtraction of (1) from (3) gives the following equation for the filtered error $e(t) = \hat{x}(t) - x(t)$:

$$e(t+1) = L(t)e(t) + \eta(t), \quad (11)$$

where $\eta(t) := K(t+1)v(t+1) - [I - K(t+1)H]w(t)$, $L(t) := [I - K(t+1)H]\Phi$.

Consider the filter (3) with the time-invariant gain (7). Then $L(t) = L$ is time-invariant. In the block matrix (5) let ν be a non-negative integer number such that

$$\sigma_{\nu+1} \leq \frac{\epsilon}{\|B\|}, B := K_eH \quad \text{for some fixed } \epsilon \in (0, 1). \quad (12)$$

For the sake of simplicity we first introduce the condition

In (2) the $(p \times n)$ matrix$H$is such that $\text{rank}[H_e] = \nu$ (13)

where $H_e = HU_1$. We emphasize that this condition requires $p \geq \nu$ and the ν columns of H_e to be linearly independent.

Lemma 3.1. *Consider the SVD (6) and introduce the block matrix* $\tilde{D}^{(2)} = \begin{vmatrix} -A \\ I_{n-\nu} \end{vmatrix}$ *and* $\tilde{D}_{(1)} = [I_\nu, A]$, $A = K_eHU_2$. *Then*

$$\|\tilde{D}^{(2)}\| = \|\tilde{D}_{(1)}\| = \|K_eH\|.$$

Proof. We have $\|\tilde{D}^{(2)}\|^2 = \sigma_1[C^{(2)}]$, $C^{(2)} = \tilde{D}^{(2),T}\tilde{D}^{(2)} = I + A^TA$. Analogously, $\|\tilde{D}_{(1)}\|^2 = \sigma_1[C_{(1)}]$, $C_{(1)} = \tilde{D}_{(1)}\tilde{D}_{(1)}^T = I + AA^T$.