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Analysis and synthesis of attractive quantum Markovian dynamics[☆]

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ABSTRACT

We propose a general framework for investigating a large class of stabilization problems in Markovian quantum systems. Building on the notions of invariant and attractive quantum subsystem, we characterize attractive subspaces by exploring the structure of the invariant sets for the dynamics. Our general analysis results are exploited to assess the ability of open-loop Hamiltonian and output-feedback control strategies to synthesize Markovian generators which stabilize a target subsystem, subspace, or pure state. In particular, we provide an algebraic characterization of the manifold of stabilizable pure states in arbitrary finite-dimensional Markovian systems, that leads to a constructive strategy for designing the relevant controllers. Implications for stabilization of entangled pure states are addressed by example.

1. Introduction

Stabilization problems have a growing significance for a variety of quantum control applications, ranging from state preparation of optical, atomic, and nano-mechanical systems to the generation of noise-protected realizations of quantum information in realistic devices (Knill, 2006; Knill, Laflamme, & Viola, 2000). Dynamical systems undergoing Markovian evolution (Alicki & Lendi, 1987; Breuer & Petruccione, 2006) are relevant for typifying irreversible quantum dynamics and present distinctive control challenges (Altafini, 2003). In particular, open-loop quantum-engineering and (approximate) stabilization methods based on dynamical decoupling cease to be viable in the Markovian regime (Lloyd & Viola, 2001; Viola, Knill, & Lloyd, 1999). Our goal in this work is to show how a wide class of Markovian stabilization problems can nevertheless be effectively treated within a general framework, provided by *invariant and attractive quantum subsystems*.

After providing the relevant technical background, we proceed to establish a first analysis result that fully characterizes the attractive subspaces for a given generator. This is done by analyzing the structure induced by the generator in the system's Hilbert space, and by invoking Krasovskii–LaSalle's invariance principle. We next explore the application of the result to stabilization problems for Markovian Hamiltonian and output-feedback control. Our approach leads to a complete characterization of the stabilizable pure states, subspaces, and subsystems as well as to constructive design strategies for the control parameters. While some partial results in this sense have been presented in Ticozzi and Viola (2008) and Viola and Ticozzi (2007), two major advances stem from the fact that all the stabilization conditions identified in the present analysis are both *necessary and sufficient* (as opposed to mostly sufficient criteria in Ticozzi and Viola (2008)) and applicable to *arbitrary finite-dimensional systems*, further extending the results for two-level systems in Ticozzi and Viola (2008). We refer to Ticozzi and Viola (2008) for a more detailed discussion of the connection between invariant, attractive, and *noiseless* subsystems, along with a thorough analysis of model robustness issues which shall not be our focus here.

2. Preliminaries and background

2.1. Quantum dynamical semigroups

Throughout our analysis, we shall consider a *finite-dimensional* quantum system $\mathcal{Q}$. Following the standard quantum-statistical mechanics formalism (Sakurai, 1994), we associate to $\mathcal{Q}$ a (separable) Hilbert space $\mathcal{H}$ over the complex field $\mathbb{C}$. Using Dirac's notation, let the vectors be represented by a *ket* $|\psi\rangle \in \mathcal{H}$, and linear functionals by a *bra*, $\langle\psi| \in \mathcal{H}^\dagger$, respectively. The inner product of $|\psi\rangle, |\varphi\rangle$ is then represented as $\langle\psi|\varphi\rangle$. Let $\mathfrak{B}(\mathcal{H})$ represent the set of linear bounded operators on $\mathcal{H}$, with $\mathfrak{H}(\mathcal{H})$

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denoting the real subspace of Hermitian operators, and $\mathbb{I}$, $\mathbb{O}$ being the identity and the zero operator, respectively. Our (possibly uncertain) knowledge of the state of $\mathcal{Q}$ is contained in a *density operator* ρ on $\mathcal{H}$, with $\rho \geq 0$ and $\text{trace}(\rho) = 1$. Density operators form a convex set $\mathfrak{D}(\mathcal{H}) \subset \mathfrak{H}(\mathcal{H})$, with one-dimensional projectors corresponding to extreme points (*pure states*, $\rho_{|\psi\rangle} = |\psi\rangle\langle\psi|$). Observables are represented by Hermitian operators in $\mathfrak{H}(\mathcal{H})$, and expectation values are computed by using the trace functional: $\mathbb{E}_\rho(X) = \text{trace}(\rho X)$. If $\mathcal{Q}$ consists of two distinguishable quantum systems $\mathcal{Q}_1$, $\mathcal{Q}_2$, the corresponding description is carried out in the tensor product space, $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$, observables and density operators being associated with Hermitian and positive-semidefinite, normalized operators on $\mathcal{H}_{12}$, respectively. The *partial trace* over $\mathcal{H}_2$ is the unique linear operator $\text{trace}_2(\cdot) : \mathfrak{B}(\mathcal{H}_{12}) \to \mathfrak{B}(\mathcal{H}_1)$, ensuring that for every $X_1 \in \mathfrak{B}(\mathcal{H}_1), X_2 \in \mathfrak{B}(\mathcal{H}_2)$, $\text{trace}_2(X_1 \otimes X_2) = X_1 \text{trace}(X_2)$. Partial trace is used to compute marginal states and partial expectations.

In the presence of either intended or unwanted couplings (such as with a measurement apparatus, or with a surrounding quantum environment), the evolution of a subsystem of interest is no longer unitary and reversible, and the formalism of *open quantum systems* is required (Alicki & Lendi, 1987; Breuer & Petruccione, 2006). A wide class of open quantum systems obey Markovian dynamics (Alicki & Lendi, 1987; Breuer & Petruccione, 2006; Gorini, Frigerio, Verri, Kossakowski, & Sudarshan, 1978; Lindblad, 1976). Let $\mathcal{I}$ denote the physical system of interest, with associated Hilbert space $\mathcal{H}_I$, $\dim(\mathcal{H}_I) = d$. Assume that we have no access or control over the system's environment, and that the dynamics in $\mathfrak{D}(\mathcal{H}_I)$ is continuous in time and described at each instant $t \geq 0$ by a Trace-Preserving Completely Positive (TPCP) linear map (Kraus, 1983). If a forward composition law is also assumed, we obtain a quantum Markov process, or Quantum Dynamical Semigroup (QDS).

Due to the trace- and positivity-preserving assumptions, a QDS is a semigroup of contractions (with respect to the Hilbert–Schmidt norm). As proven in Lindblad (1976) and Gorini, Kossakowski, and Sudarshan (1976), the Hille–Yoshida generator for the semigroup exists and can be cast in the following canonical form $\dot\rho_t = \mathcal{L}(\rho_t)$, where

$$\mathcal{L}(\rho_t) = -\frac{i}{\hbar}[H, \rho_t] + \sum_{k=1}^{p}\Big(L_k\rho_t L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k,\ \rho_t\}\Big). \tag{1}$$

The *effective Hamiltonian* H and the *noise operators* L_k (also known as "Lindblad operators") completely specify the dynamics, including the effect of the Markovian environment. In general, H is equal to the Hamiltonian for the isolated, free evolution of the system, H_0, plus a correction, H_L, induced by the coupling to the environment (aka "Lamb shift"). The non-Hamiltonian terms $\mathcal{D}(L_k, \rho(t))$ in (1) account for the non-unitary character of the dynamics, specified by noise operators $\{L_k\}$.

In principle, the form of the generator $\mathcal{L}$ may be rigorously derived from a Hamiltonian model for the joint system-environment dynamics under appropriate limiting conditions (the so-called "singular coupling limit" or the "weak coupling limit," respectively (Alicki & Lendi, 1987; Breuer & Petruccione, 2006)). In most practical situations, however, such a derivation is unfeasible, since the full microscopic Hamiltonian is unavailable. A Markovian generator of the form (1) is then postulated on a phenomenological basis, and available knowledge about the noise effects used to infer a specific set of operators L_k (not necessarily orthogonal or complete) in (1). Physically, each such operator may be associated to a distinct *noise channel*, by which information irreversibly leaks from the system to the environment.

2.2. Quantum subsystems: Invariance and attractivity

Quantum subsystems are the basic building blocks for describing *composite systems* in quantum mechanics (Sakurai, 1994), and provide a general framework for scalable quantum information engineering in physical systems. In fact, the so-called *subsystem principle* (Knill, 2006; Knill et al., 2000) states that any "faithful" representation of information in a quantum system requires to specify a subsystem that encodes the desired information. Many of the control tasks considered in this paper are motivated by the need for strategies to create and maintain quantum information in open quantum systems. A definition of quantum subsystem suitable to our context is the following:

Definition 1 (*Quantum Subsystem*). A *quantum subsystem* $\mathcal{S}$ of a system $\mathcal{I}$ defined on $\mathcal{H}_I$ is a quantum system whose Hilbert space is a tensor factor $\mathcal{H}_S$ of a subspace $\mathcal{H}_{SF}$ of $\mathcal{H}_I$,

$$\mathcal{H}_I = \mathcal{H}_{SF} \oplus \mathcal{H}_R = (\mathcal{H}_S \otimes \mathcal{H}_F) \oplus \mathcal{H}_R, \tag{2}$$

for some co-factor $\mathcal{H}_F$ and remainder space $\mathcal{H}_R$.[1] The set of linear operators on $\mathcal{S}$, $\mathcal{B}(\mathcal{H}_S)$, is isomorphic to the (associative) subalgebra $\mathcal{B}(\mathcal{H}_I)$ of operators of the form $X_I = X_S \otimes \mathbb{I}_F \oplus \mathbb{O}_R$.

Let $n = \dim(\mathcal{H}_S), f = \dim(\mathcal{H}_F), r = \dim(\mathcal{H}_R)$, and let $\{|\phi_j^S\rangle\}_{j=1}^n$, $\{|\phi_k^F\rangle\}_{k=1}^f$, $\{|\phi_l^R\rangle\}_{l=1}^r$ be orthonormal bases for $\mathcal{H}_S$, $\mathcal{H}_F$, $\mathcal{H}_R$, respectively. The decomposition (2) is then naturally associated with the following basis for $\mathcal{H}_I$:

$$\{|\varphi_m\rangle\} = \{|\phi_j^S\rangle \otimes |\phi_k^F\rangle\}_{j,k=1}^{n,f} \cup \{|\phi_l^R\rangle\}_{l=1}^r.$$

This induces a block structure for matrices acting on $\mathcal{H}_I$, which will be used to represent density matrices, observables, and generators:

$$X = \left(\begin{array}{c|c} X_{SF} & X_P \\ \hline X_Q & X_R \end{array}\right), \tag{3}$$

where, in general, $X_{SF} \neq X_S \otimes X_F$. We denote by Π_{SF} the projector onto $\mathcal{H}_{SF}$, whereas $\bar\Pi_{SF} : \mathcal{H}_I \to \mathcal{H}_S$ represents its reduction $\bar\Pi_{SF} = \left(\begin{array}{c|c} \mathbb{I}_{SF} & 0 \end{array}\right)$. In this paper, we study Markov dynamics of a quantum system $\mathcal{I}$ with a *given* decomposition of the associated Hilbert space of the form (2), with respect to the subsystem $\mathcal{S}$ associated to $\mathcal{H}_S$. By describing the dynamics in the Schrödinger picture (*i.e.*, with evolving states and time-invariant observables), the first step is to specify whether $\mathcal{I}$ has been properly initialized in a state which faithfully represents a state of $\mathcal{S}$, and what is the structure of such states.

Definition 2 (*State Initialization*). The system $\mathcal{I}$ with state $\rho \in \mathfrak{D}(\mathcal{H}_I)$ is *initialized in* $\mathcal{H}_S$ *with state* $\rho_S \in \mathfrak{D}(\mathcal{H}_S)$ if the blocks of ρ satisfy:

(i) $\rho_{SF} = \rho_S \otimes \rho_F$ for some $\rho_F \in \mathfrak{D}(\mathcal{H}_F)$;
(ii) $\rho_P = 0$, $\rho_R = 0$.

We denote by $\mathfrak{I}_S(\mathcal{H}_I)$ the set of states that satisfy (i)–(ii) for some ρ_S.

Condition (ii) guarantees that $\bar\rho_S = \text{trace}_F(\Pi_{SF}\rho\Pi_{SF}^\dagger)$ is a valid (normalized) state of $\mathcal{S}$, while (i) ensures that measurements or dynamics affecting $\mathcal{H}_F$ have no effect on the state in $\mathcal{H}_S$. In the case of a simple subspace decomposition, we have $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$, with $\mathcal{H}_F \approx \mathbb{C}$, so that (i) becomes trivial. $\mathfrak{I}_S(\mathcal{H}_I)$ then indicates the set of states satisfying (ii) for the given subspace.

We now proceed to characterize in which sense, and under which conditions, a quantum subsystem may be defined as invariant. Recall that a set $\mathcal{W}$ is said to be *invariant* for a dynamical

[1] Note that in control theory a splitting of the above form is referred to as identifying a "submanifold", while "subsystem" usually refer to dynamical properties.