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Stability tests and stabilization for piecewise linear systems based on poles and zeros of subsystems ☆

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Abstract

This paper provides several stability tests for piecewise linear systems and proposes a method of stabilization for bimodal systems. In particular, we derive an explicit and exact stability test for planar systems, which is given in terms of coefficients of transfer functions of subsystems. Restricting attention to the bimodal and planar case, we show simple stability tests. In addition, we drive a necessary stability condition and a sufficient stability condition for higher-order and bimodal systems. They are given in terms of the eigenvalue loci and the observability of subsystems. All the stability tests provided in this paper are computationally tractable, and our results are applied to the stabilizability problem. We confirm the exactness and effectiveness of our approach by illustrative examples.

Keywords: Hybrid dynamical systems; Piecewise linear systems; Stability analysis; Stabilization; Poles; Zeros

1. Introduction

Hybrid control has received much attention in the area of control system design, since we have many practical control applications which contain both continuous-time dynamics and logical or switching elements. There have been a lot of mathematical models proposed to represent behaviors of hybrid control systems. One of the typical models is the piecewise linear (PWL) system. The system consists of some pairs of linear time-invariant dynamics and a cell which is a piece of a partition of the state space, and the state evolves along the dynamics associating with the cell in which the state exists. The class of PWL systems is one of the fundamental classes of hybrid dynamical systems, since the hybrid dynamics is quiet simple compared with those of other classes of hybrid control systems. Study on PWL systems is therefore important as a first step to establish hybrid control theory. Actually, stability analysis of PWL systems is required for stability analysis of piecewise affine systems in a neighborhood of the origin.

In the last decade, many results have been obtained on stability for several classes of hybrid dynamical systems (see Decarlo, Branicky, Petersson, & Lennartson, 2000; Liberzon, 2003; Lygeros, Johansson, Simic, Zhang, & Sastry, 2003 and the references therein). Most of the results are extensions of Lyapunov's theorem, where we need to show the existence of a Lyapunov function which guarantees the stability. The Lyapunov methods provide not only sufficient conditions but also necessary conditions for stability under hybrid natures. Actually, the converse theorems ensure the existence of a Lyapunov function when the system is asymptotically stable (Dayawansa & Martin, 1999; Michel & Hu, 1999; Molchanov & Pyatnitskii, 1986; Ye, Michel, & Hou, 1998).

In spite of the recent progress, there still remain fundamental problems on stability for hybrid control systems to be clarified. A major problem is how to check the stability exactly. We must restrict available classes of Lyapunov functions within a class of piecewise quadratic functions (Gonçalves, Megretski, & Dahleh, 2003; Pettersson & Lennartson, 2002; Rantzer & Johansson, 2000) or a class of sums of squares (Prajna & Papachristodoulou, 2003; Spanos & Gonçalves, 2004) to give

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systematic ways of finding the Lyapunov functions. This makes the stability conditions conservative. In fact, we cannot check the stability even for the class of PWL systems exactly. In particular, for any bimodal PWL system with an unstable subsystem, there exists no piecewise quadratic Lyapunov function even if the origin is stable or the *S*-procedure is applied (van der Schaft & Schmacher, 2000), though the *S*-procedure is sometimes efficient for checking the stability of general PWL systems with unstable subsystems. We therefore need a new approach to get a less conservative stability condition or hopefully to derive a necessary and sufficient stability condition.

To this end, we here discuss the stability problem for PWL systems from a different perspective. Instead of using Lyapunov's theory, we investigate behavior of PWL systems directly to get stability tests which are computationally tractable. Recently, direct analysis of behaviors of hybrid states has led to exact stability tests for certain classes of switched systems. Xu and Antsaklis (2000) derived necessary and sufficient conditions for stabilizability of a class of planar and linear switched systems through an investigation of behaviors of the systems. In Boscain (2002), necessary and sufficient stability conditions were provided for a class of planar switched systems with arbitrary switching. None of the two papers treat any extensions of Lyapunov's theorem. Note that the two classes of switched systems are quiet different from the class of PWL systems, since the switching in the two classes of switched systems does not depend on the continuous state. Indeed, the switching is an input to the systems in Xu and Antsaklis (2000) and the switching takes place arbitrarily in Boscain (2002). Hence, we require yet different investigation from each of the classes of switched systems.

This paper provides several stability tests for PWL systems and proposes a method of stabilization for bimodal systems. In particular, we derive an explicit and exact stability test for planar systems, which is given in terms of coefficients of transfer functions of subsystems. Restricting attention to the bimodal and planar case, we show simple stability tests. In addition, we drive a necessary stability condition and a sufficient stability condition for higher-order and bimodal systems. They are given in terms of the eigenvalue loci and the observability of subsystems. All the stability tests provided in this paper are computationally tractable. Our results are then applied to the stabilizability problem. We confirm the exactness and effectiveness of our approach by illustrative examples.

This paper is organized as follows. Section 2 gives a basic setup for representing a class of PWL systems. Section 3 is devoted to stability analysis based on behavior of systems. We give an explicit and exact stability test for planar PWL systems in terms of coefficients of transfer functions of subsystems in Section 4. Section 5 considers bimodal case, and we provide simple stability tests for planar systems and derive a necessary condition and a sufficient condition for stability of higher-order systems. In Section 6, we discuss the stabilizability problem based on the sufficient stability condition. Most of proofs are collected in appendices.

In this paper, we will use the following notation. The symbols $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{R}_+$ represent the set of integers, the set of real numbers, and the set of positive real numbers, respectively. The symbols $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ stand for the set of all n-dimensional real column vectors and the set of all $n \times m$ real matrices, respectively.

2. Piecewise linear systems

We consider a class of PWL systems represented by

$$\dot{x} = f(x) := \begin{cases} A_1 x & \text{if } x \in \mathbb{S}_1, \\ A_2 x & \text{if } x \in \mathbb{S}_2, \\ \vdots & \vdots \\ A_m x & \text{if } x \in \mathbb{S}_m, \end{cases} \tag{1}$$

where $A_i \in \mathbb{R}^{n\times n}$ and $\mathbb{S}_i$ are convex cone of the form

$$\mathbb{S}_i := \{x \in \mathbb{R}^n | C_i x \geqslant 0\}, \tag{2}$$

with $C_i \in \mathbb{R}^{n\times n}$ $(i = 1, \ldots, m)$. A matrix A_i may be equal to A_j $(j \neq i)$ as seen in Fig. 1. The vector field $f(x)$ may be discontinuous on the boundary $\partial\mathbb{S}_i$. The solution from a given initial state x_0 is denoted by $x(t, x_0)$ where the initial time is always set 0. This paper basically focuses on two subclasses of PWL systems, called the planar case and the bimodal case. The planar case implies $n = 2$ and $m \geqslant 2$. The bimodal case means $m = 2$ and $n \geqslant 2$.

We here use two notions, called memoryless and well-posedness, to clarify the class of systems treated in this paper.

First, system (1) is said to be *memoryless*, if all the following conditions hold:

$$\mathbb{S}_i \neq \mathbb{R}^n \quad \forall i, \tag{3}$$

$$\operatorname{int} \mathbb{S}_i \neq \emptyset \quad \forall i, \tag{4}$$

$$\cup_{i=1}^m \mathbb{S}_i = \mathbb{R}^n, \tag{5}$$

$$\operatorname{int}(\mathbb{S}_i \cap \mathbb{S}_j) = \emptyset \quad \forall (i, j), \ (i \neq j). \tag{6}$$

It is clear that (3)–(6) are quite natural and hence they are not restrictive for systems with memoryless nonlinearities. Note that both (3) and (4) hold, if $\det C_i \neq 0$ for all i. In addition, every bimodal system can be represented by

$$\dot{x} = \begin{cases} A_1 x & \text{if } cx \geqslant 0, \\ A_2 x & \text{if } cx \leqslant 0, \end{cases} \tag{7}$$

where $c(\neq 0) \in \mathbb{R}^{1\times n}$, if the system is memoryless.

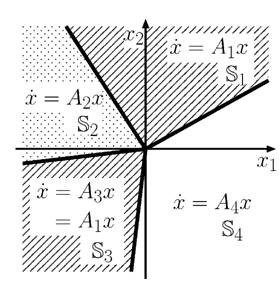

Fig. 1. Planar and multi-modal piecewise linear model.