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Brief paper

Robust stability analysis of linear time-delay systems by Lambert W function: Some extreme point results[☆]

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Abstract

Robust stability conditions are derived for linear time-delay systems using Lambert W function. The characteristic quasi-polynomials of the systems are assumed to be factorized. It is proven that if uncertainties in the coefficients of the quasi-polynomial are set in appropriate regions in the complex plane, we can enjoy extreme point results: finite number of stability checks at some points of the boarder of the regions suffice. The strength of Lambert W function approach lies in the fact that the function is implemented on some standard software packages such as Mathematica, Maple or Matlab which afford to compute the function value very easily. The above two points make the stability test for the class of uncertain time-delay systems quite practical.

Keywords: Linear time-delay systems; Quasi-polynomials; Lambert W function; Robust stability; Extreme point results

1. Introduction

Stability robustness is one of central concerns for control of time-delay systems and considerable amount of results on the topic are currently available. If the focus is confined to linear time-delay systems, two different approaches are mainly taken: time domain approach and frequency domain one. For the time domain approach, Lyapunov–Krasovskii functionals are employed, leading to various LMI conditions (Gu, 2001; Gu, Kharitonov, & Chen, 2003; Niculescu, 2001; Park, 1999). While they are mostly sufficient conditions for the stability, some can give necessary and sufficient conditions using specific Lyapunov–Krasovskii functionals (Gu et al., 2003; Kharitonov & Zhabko, 2003). As the frequency domain approach, frequency sweeping tests (Chen & Niculescu, 2004; Gu et al., 2003) and the Edge Theorem (Fu, Olbrot, & Polis, 1989; Gu et al., 2003; Niculescu, 2001) are known. In Chen and Niculescu (2004), a sufficient condition for quasi-polynomials with commensurate delays to be robustly stable independent of delay has been derived. In Wang, Hu, and Kupper (2004), an algorithm to check robust stability of a polytope of quasi-polynomials with the help of the Edge Theorem has been proposed. The result gives an exact answer to the stability problem, yet the computational complexity surfaces as the number of edge quasi-polynomials grows.

An alternative way in the frequency domain approach is to invoke Lambert W function (Asl & Ulsoy, 2000; Hwang & Cheng, 2005). The function has found many applications in a variety of science and engineering disciplines (see e.g., Corless, Gonnet, Hare, Jeffrey, & Knuth, 1996). In Asl and Ulsoy (2000), the function is used for stability analysis of scalar linear time-delay systems and of linear delay systems having only a delay term. The function is also used in Hwang and Cheng (2005) for analysis of fractional-order time-delay systems and cautions are given there about its application to such systems.

In this paper, we apply the Lambert W function method to robust stability analysis of linear time-delay systems with single delay. The advantages of the function consist of the facts that it enables to express the characteristic roots of the systems explicitly and to give non-conservative analysis results. Making the best of these points, we derive necessary and sufficient

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conditions for linear time-delay systems to be robustly stable dependent of delay. A proviso is, however, that system characteristic quasi-polynomials are factorized so that the analysis boils down to studying scalar systems. This may narrow the applicability of the obtained results, but would be, to the authors' belief, the price paid to obtain the above advantageous points. Some triangularizability conditions serve for the factorization. It should be stressed that the conditions are extreme point results: when uncertainties of the coefficients of the systems are expressed as suitable bounded sets, the conditions require only information on extreme points of the sets of the uncertainties. It should be also noted that commonly available software such as Mathematica, Maple or Matlab which implement calculator of the Lambert W function enables to check the robust stability easily.

The organization of this paper is as follows. In the next section, we give introductory expositions of the Lambert W function with its definition and a key property. The proof of the property is relegated to Appendix A. Section 3 gives the main results. Necessary and sufficient robust stability conditions are derived for linear time-delay systems with box-type uncertainty in the non-delay term coefficient and sector-type uncertainty in the delay term one. A numerical example of the obtained results is illustrated in Section 4 and Section 5 concludes the paper.

2. Lambert W function

Lambert W function is defined as the function satisfying

$$W(z)\mathrm{e}^{W(z)} = z, \tag{1}$$

where $W : \mathbf{C} \to \mathbf{C}$ (Corless et al., 1996). W maps z-plane to w-plane, i.e. $w = W(z)$, and can be expressed as

$$z = a + jb, \quad w = \xi + j\eta. \tag{2}$$

Then substituting (2) to (1) gives

$$a = \mathrm{e}^{\xi}(\xi\cos\eta - \eta\sin\eta), \quad b = \mathrm{e}^{\xi}(\eta\cos\eta + \xi\sin\eta). \tag{3}$$

Without loss of generality, we can restrict the argument of z-variable to $(-\pi, \pi]$.

Remark 1. It is noted that since (1) includes no complex parameters, the range of the Lambert W function is symmetric with respect to the real axis.

Lambert W function is a multi-valued function, i.e., it has infinitely many branches. We express the branches as W_k, $k = 0, \pm 1, \ldots, \pm\infty$. Especially, W_0 is said to be the principal branch. W_k, $k = 0, \pm 1, \ldots, \pm\infty$ are single-valued functions, respectively. Fig. 1 shows the range of each branch. Among them, W_0, W_1 and W_{-1} are mainly related in later discussions.

The branch cut of W_0 is defined by $\{z| -\infty < \mathrm{Re}(z) \leqslant -1/e,\ \mathrm{Im}(z) = 0\}$, while that of the other branches are defined by $\{z \mid -\infty < \mathrm{Re}(z) \leqslant 0,\ \mathrm{Im}(z) = 0\}$.[1] Figs. 2–4 show the mapping of z-plane by W_0, W_1 and W_{-1}, respectively.

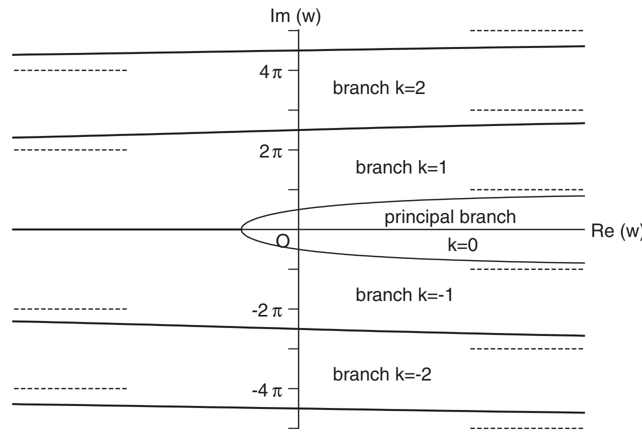

Fig. 1. Ranges of the branches of Lambert W function.

For W_k, $k \neq \pm 1$, the image of the argument $+\pi$ of the branch cut in z-plane corresponds to the boundary of W_{k+1} in w-plane, and the argument $-\pi$ corresponds to the boundary of W_{k-1}. For W_1, the argument $+\pi$ of the branch cut corresponds to that of W_2, while the argument $-\pi$ of the part of the branch cut, $\{z \mid -\infty < \mathrm{Re}(z) \leqslant -1/e,\ \mathrm{Im}(z) = 0\}$, corresponds to that of W_0. The image of the remained $\{z \mid -1/e < \mathrm{Re}(z) \leqslant 0,\ \mathrm{Im}(z) = 0\}$ by W_1 corresponds to that of W_{-1} (Fig. 3). Similar configuration occurs among W_{-1}, W_{-2}, and W_0 (Fig. 4). In these three figures, markings A–F indicate the correspondence between the two planes.

Remark 2. If W_k, $k=0, \pm 1, \ldots, \pm\infty$ map a curve not crossing the corresponding branch cut, each of W_k, $k = 0, \pm 1, \ldots, \pm\infty$ becomes a homeomorphism, a mapping which, as well as whose inverse, is both continuous and one to one.

The following property of the Lambert W function is very important throughout the rest of discussion.

Lemma 3. *For arbitrary $z \in \mathbf{C}$,*

$$\max\{\mathrm{Re}(W_k(z))|k = 0, \pm 1, \ldots, \pm\infty\} = \mathrm{Re}(W_0(z))$$

is satisfied.[2]

Though this property might have been proven in much earlier literature, we give a proof in Appendix A to make the discussion self-contained. Note that Lemmas 11 and 12 in Appendix A, which are used to show the above lemma, are also cited in the body of the paper subsequently. Here, however, we give only an intuitive observation leading to Lemma 3. Consider an image of a circle

$$z = r\mathrm{e}^{j\theta}, \quad \theta \in (-\pi, \pi] \tag{4}$$

[1] Though this definition is different from that in Corless et al. (1996), we believe it is more suitable so far as W_1 and W_{-1} are concerned.

[2] For ease of argument, we make a sort of compactification, i.e., we regard both W_{∞} and $W_{-\infty}$ as fixed mappings.