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Brief paper

$\mathcal{L}_2$ gain analysis for a class of switched systems[☆]

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ABSTRACT

This paper considers the problem of disturbance tolerance/rejection for a family of linear systems subject to actuator saturation and $\mathcal{L}_2$ disturbances. For a given set of linear feedback gains, a given switching scheme and a given bound on the $\mathcal{L}_2$ norm of the disturbances, conditions are established in terms of linear or bilinear matrix inequalities under which the resulting switched system is bounded state stable, that is, trajectories starting from a bounded set will remain inside the set or a larger bounded set. With these conditions, both the problem of assessing the disturbance tolerance/rejection capability of the closed-loop system and the design of feedback gain and switching scheme can be formulated and solved as constrained optimization problems. Disturbance tolerance is measured by the largest bound on the disturbances for which the trajectories from a given set remain bounded. Disturbance rejection is measured by the restricted $\mathcal{L}_2$ gain over the set of tolerable disturbances. In the event that all systems in the family are identical, the switched system reduces to a single system under a switching feedback law. It will be shown that such a single system under a switching feedback law has stronger disturbance tolerance/rejection capability than a single linear feedback law can achieve.

1. Introduction

The literature on analysis and design of switched systems has been growing rapidly in recent years (see, for example, Branicky (1994), Cheng (2005), DeCarlo, Branicky, Pettersson, and Lennartson (2000), Liberzon and Morse (1999), Pettersson (1999), Pettersson and Lennartson (2001), Sun and Ge (2005), Wicks, Peleties, and DeCarlo (1998) and Xi, Feng, Jiang, and Cheng (2003) and the references therein). Motivated by the results reported in this literature, we consider in this paper the following family of linear systems subject to input saturation and disturbances,

$$\begin{cases} \dot{x} = A_i x + B_i \mathrm{sat}(u) + E_i w, \\ z = C_i x, \quad i \in I_N := \{1, 2, \ldots, N\}, \end{cases} \tag{1}$$

where $x \in \mathbf{R}^n, u \in \mathbf{R}^m, z \in \mathbf{R}^p$ are respectively the state, input and output of the system, $w \in \mathbf{R}^q$ represents the disturbances, and $\mathrm{sat} : \mathbf{R}^m \to \mathbf{R}^m$ is the vector valued standard saturation function $\mathrm{sat}(u) = \left[\mathrm{sat}(u_1) \quad \mathrm{sat}(u_2) \quad \cdots \quad \mathrm{sat}(u_m)\right]^{\mathrm{T}}$, $\mathrm{sat}(u_i) = \mathrm{sign}(u_i)\min\{|u_i|, 1\}$. A switched system then results by defining a controller/supervisor which chooses one of the systems at each time instant based on the measurement of the state and according to an index function, say, $i = \sigma(x)$. A typical form of the index function is $\sigma(x) = i$ for $x \in \Omega_i$ with $\cup_{i=1}^{N} \Omega_i = \mathbf{R}^n$. Thus, the control design involves the construction of both feedback gains for individual systems and the index function so that the resulting switched system possesses certain desired performances.

In the absence of the disturbances w, a basic design objective is the local asymptotic stability of the resulting switched system with as large a domain of attraction as possible. By utilizing some techniques in dealing with actuator saturation (Hu & Lin, 2001) and the form of the largest region index function proposed by Pettersson (2003, 2004, 2005), we recently proposed a method for the design of the individual feedback gains and the index function that result in a locally asymptotically stable switched system Lu and Lin (2008). The design is formulated and solved as a constrained optimization problem with the objective of enlarging the domain of attraction of the resulting stable equilibrium at the origin. It was shown by numerical examples that such a design may result in a domain of attraction larger than that of a switched system, designed without taking actuator saturation into account.

In this paper, we will carry out an analysis of, and design for, the disturbance tolerance/rejection capability of the switched system resulting from the family of systems (1). We will restrict ourselves

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to a class of disturbances whose energies are bounded by a given value, i.e.,

$$\mathcal{W}_\alpha^2 := \left\{ w : \mathbf{R}_+ \to \mathbf{R}^q : \int_0^\infty w^{\mathrm{T}}(t)w(t)\mathrm{d}t \leq \alpha \right\}, \tag{2}$$

for some positive number α. For a given set of linear feedback gains, a given index function and a given value of α, conditions will be established in terms of linear or bilinear matrix inequalities under which the resulting switched system is bounded state stable. A system is said to be bounded state stable if its trajectories starting from a bounded set will remain inside the set or a larger bounded set. With these conditions, both the problem of assessing the disturbance tolerance/rejection capability of the closed-loop system and the design of feedback gain and switching scheme can be formulated and solved as constrained optimization problems. Disturbance tolerance is measured by the largest bound on the energy of the disturbance, α^*, for which the trajectories from a given set remain bounded. Disturbance rejection is measured by the restricted $\mathcal{L}_2$ gain over $\mathcal{W}_{\alpha^*}^2$.

An interesting special class of the systems we consider in this paper is the case when all the systems in (1) are identical. In this case, the switched system reduces to a single system under a switching linear feedback law. It will be shown that for a single linear system of the form (1), a switching feedback law will result in stronger disturbance tolerance/rejection capability than a single linear feedback law of Fang, Lin, and Hu (2004) and Fang, Lin, and Shamash (2006). The $\mathcal{L}_2$ gain analysis and design for linear systems under actuator saturation has been studied by several authors. A small sample of their works include Chitour, Liu, and Sontag (1995), Fang et al. (2004, 2006), Hindi and Boyd (1998), Hu and Lin (2001), Lin (1997), Nguyen and Jabbari (1999) and Xie, Wang, Hao, and Xie (2004). In particular, in our recent work (Fang et al., 2004, 2006), we considered the $\mathcal{L}_2$ gain analysis and design for a linear system under actuator saturation. The disturbance tolerance capability of the closed-loop system under a given feedback law was assessed, and the linear feedback law that results in a minimized restricted $\mathcal{L}_2$ gain was designed.

The remainder of this paper is organized as follows. In Section 2, we state our problem and recall some preliminary materials that will be needed in the development of the results of this paper. Section 3 establishes bounded state stability conditions. Disturbance tolerance and disturbance rejection are addressed in Sections 4 and 5, respectively. Simulation results are presented in Section 6. Section 7 concludes the paper.

2. Problem statement and preliminaries

For the family of systems (1), we would like to design a linear feedback law for each individual system in the family and an index function such that the resulting switched system possesses a high degree of disturbance tolerance and a high level of disturbance rejection capabilities.

We will adopt the switching strategy of Pettersson (2003, 2004). Such a switching strategy is defined based on some appropriately chosen symmetric matrices $Q_i \in \mathbf{R}^{n\times n}$, $i \in I_N$. More specifically, at a given state x, the subsystem i will be activated if the quadratic function $x^{\mathrm{T}}Q_ix$ is greater or equal to any other $x^{\mathrm{T}}Q_jx$, $j \neq i$. More specifically, this switching scheme is defined by the following index function Pettersson (2003, 2004), referred to as the largest region function,

$$i(x) = \arg\left\{\max_{i\in I_N} x^{\mathrm{T}}Q_ix\right\}. \tag{3}$$

Based on the matrices Q_i's, we define the following sets

$$\Omega_i = \{x \in \mathbf{R}^n | x^{\mathrm{T}}Q_ix \geq 0\}, \quad i \in I_N,$$
$$\Omega_{i,j} = \{x \in \mathbf{R}^n | x^{\mathrm{T}}Q_jx = x^{\mathrm{T}}Q_ix \geq 0\}, \quad i \in I_N, j \in I_N.$$

Then, a well-defined switched system must satisfy the following properties:

- Covering property: $\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N = \mathbf{R}^n$;
- Switching property: $\Omega_{i,j} \subseteq \Omega_i \cap \Omega_j$, $i \in I_N$, $j \in I_N$.

The first condition says that there are no regions in the state space where none of the subsystem is activated. The second condition, which is automatically satisfied by this choice of Ω_i and $\Omega_{i,j}$, means that a switch from subsystem i to j occurs only for states where the regions Ω_i and Ω_j are adjacent. Consequently, switching occurs on the switching surface $x^{\mathrm{T}}Q_ix = x^{\mathrm{T}}Q_jx$. The following regarding the covering property was established in Pettersson (2003, 2004).

Lemma 1 (*Covering property*). *If for every* $x \in \mathbf{R}^n$,

$$\theta_1x^{\mathrm{T}}Q_1x + \theta_2x^{\mathrm{T}}Q_2x + \cdots + \theta_Nx^{\mathrm{T}}Q_Nx \geq 0, \tag{4}$$

where $\theta_i > 0$, $i \in I_N$, *then* $\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N = \mathbf{R}^n$.

3. Bounded state stability

We recall a tool from Hu and Lin (2001) for expressing a saturated linear feedback $u = \mathrm{sat}(Fx)$ on the convex hull of a mixture of the unsaturated control inputs and the auxiliary inputs. For an $F \in \mathbf{R}^{m\times n}$, let $\mathcal{L}(F) = \left\{x \in \mathbf{R}^n : |f_ix| \leq 1,\ i \in I_m\right\}$, where f_i represents the ith row of matrix F. We note that $\mathcal{L}(F)$ represents the region in $\mathbf{R}^n$ where Fx does not saturate.

Also, let $\mathcal{V}$ be the set of $m\times m$ diagonal matrices whose diagonal elements are either 1 or 0. There are 2^m elements in $\mathcal{V}$. Suppose these elements of $\mathcal{V}$ are labeled as D_s, $s \in I_{2^m}$. Denote $D_s^- = I - D_s$. Clearly, $D_s^- \in \mathcal{V}$ if $D_s \in \mathcal{V}$. The following lemma is adopted from Hu and Lin (2001).

Lemma 2. *Let* $F, H \in \mathbf{R}^{l\times n}$. *Then, for any* $x \in \mathcal{L}(H)$,

$$\mathrm{sat}(Fx) \in \mathrm{co}\left\{D_sFx + D_s^-Hx, s \in I_{2^m}\right\},$$

where co *stands for the convex hull.*

For a positive definite matrix $P \in \mathbf{R}^{n\times n}$ and a scalar $\rho > 0$, we define $\mathcal{E}(P,\rho) := \left\{x \in \mathbf{R}^n : x^{\mathrm{T}}Px \leq \rho\right\}$. The following theorem characterizes the bounded state stability of the switched system that results from the family of systems (1) and the switching scheme (3).

Theorem 1. *Consider system (1). If there exist* $P_i > 0, \xi > 0, Q_i = Q_i^{\mathrm{T}}, F_i \in \mathbf{R}^{m\times n}, H_i \in \mathbf{R}^{m\times n}, \vartheta_i \geq 0, \theta_i > 0$ *and* $\eta_{i,j}$ *such that*

1. $(A_i+B_i(D_sF_i+D_s^-H_i))^{\mathrm{T}}P_i+P_i(A_i+B_i(D_sF_i+D_s^-H_i))+\frac{1}{\xi}P_iE_iE_i^{\mathrm{T}}P_i+\vartheta_iQ_i \leq 0$, $s \in I_{2^m}$, $i \in I_N$,
2. $P_i = P_j + \eta_{i,j}(Q_j - Q_i)$, $i \in I_N, j \in I_N$,
3. $\theta_1Q_1 + \theta_2Q_2 + \cdots + \theta_NQ_N \geq 0$,

and $\mathcal{E}(P_i, 1+\alpha\xi) \cap \Omega_i \subset \mathcal{L}(H_i)$, $i \in I_N$, *then every trajectory of the closed-loop system that starts from inside of* $\cap_{i=1}^N(\mathcal{E}(P_i,1)\cap\Omega_i)$ *will remain inside of* $\cap_{i=1}^N(\mathcal{E}(P_i,1+\alpha\xi)\cap\Omega_i)$ *for every* $w \in \mathcal{W}_\alpha^2$, *as long as no sliding motion occurs or sliding motions only occur along switching surfaces with the corresponding* $\eta_{i,j} \geq 0$. *If the condition* $\mathcal{E}(P_i, 1+\alpha\xi)\cap\Omega_i \subset \mathcal{L}(H_i)$ *is replaced with* $\mathcal{E}(P_i,\alpha\xi)\cap\Omega_i \subset \mathcal{L}(H_i)$, *then any trajectory starting from the origin will remain inside the region* $\cap_{i=1}^N(\mathcal{E}(P_i,\alpha\xi)\cap\Omega_i)$ *for every* $w \in \mathcal{W}_\alpha^2$ *as long as no sliding motion occurs or sliding motions only occur along switching surfaces with the corresponding* $\eta_{i,j} \geq 0$.