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Brief paper

Convex invariant sets for discrete-time Lur'e systems[☆]

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ABSTRACT

In this paper, a method to estimate the domain of attraction of a class of discrete-time Lur'e systems is presented. A new notion of invariance, denoted *LNL*-invariance, is introduced. An algorithm to determinate the largest *LNL*-invariant set for this class of systems is proposed. Moreover, it is proven that the *LNL*-invariant sets provided by this algorithm are polyhedral convex sets and constitute an estimation of the domain of attraction of the non-linear system. It is shown that any contractive set for the Lur'e system is contained in the *LNL*-invariant set obtained applying the results of this paper. Two illustrative examples are given.

1. Introduction

The importance of Lur'e systems in the context of control theory stems from the fact that different control schemes appearing in practical applications can be formulated using the Lur'e systems structure (Arcak & Teel, 2002; Chu, Huang, & Wang, 2001; Slotine & Li, 1991; Wada, Ikeda, Ohta, & Siljak, 1989). The particular case of saturation non-linearity is widely treated in literature; see for example Hu and Lin (2001) and Tarbouriech and Gomes Da Silva Jr. (1997).

The stability analysis of a Lur'e system can be done, for example, by means of Popov and circle criterions (see Haddad, Kapila, and Chellabonia (1996), Khalil (2002), Loparo and Blankenship (1978), Vidyasagar (1993) and Weissenberger (1968)). Particular approaches are available for Lur'e systems with piecewise affine non-linearities. In this case, the domain of attraction can be estimated by means of a piecewise quadratic Lyapunov function (Johansson & Rantzer, 1998). Also, a novel result to deal with this class of Lur'e systems can be found in Hu, Huang, and Lin (2004), where a procedure to compute invariant ellipsoids is presented. Another relevant contribution to this research field is presented in Rakovic, Grieder, Kvasnica, Mayne, and Morari (2004).

In this paper, we consider Lur'e systems in which the non-linearity appearing in the feedback path has a piecewise affine nature. A new notion of invariance (*LNL*-invariance) is presented in this work. This new concept generalizes the notion of *SNS*-invariance introduced in Alamo, Cepeda, Limon, and Camacho (2006) for saturated systems. Based on its geometrical properties, a simple algorithm to obtain the largest *LNL*-invariant set is proposed. *LNL*-invariance is a more conservative concept than traditional invariance but its geometrical properties allow us to obtain a polyhedric estimation of the domain of attraction of the non-linear system. It is shown that any contractive set for the Lur'e system is an *LNL*-invariant set which is included in the obtained estimation of the domain of attraction.

The paper is organized as follows. In Section 2 the class of piecewise-affine discrete-time Lur'e systems considered is presented. In Section 3 some geometrical properties are given. The new notion of *LNL*-invariance is introduced in Section 4. This concept is used in Section 5 to estimate the domain of attraction of the non-linear system. Two illustrative examples are given in Section 6. The paper draws to a close with a section of conclusions.

2. Problem statement

Consider the following discrete-time Lur'e system:

$$\begin{cases} x_{k+1} = Ax_k - B\phi(y_k) \\ y_k = Fx_k, \end{cases} \quad (1)$$

where $x_k \in \mathbb{R}^n$ represents the state vector and $y_k = Fx_k \in \mathbb{R}$ the output of the system. The non-linear function $\phi(\cdot)$ is assumed to satisfy the following conditions:

(i) $\phi(y)$ is piecewise-affine.
(ii) $\phi(y)$ is a continuous odd function.
(iii) $\phi(y)$ is concave in $\mathbb{R}^+$.

The following property characterizes all the functions $\phi(\cdot)$ that satisfy previous assumptions.

Property 1 (Hu et al. (2004)). *The piecewise-affine function $\phi(y)$ is concave in $\mathbb{R}^+$ if and only if it can be expressed as*

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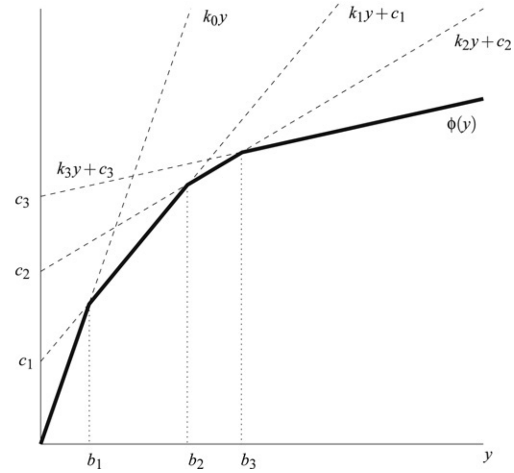

Fig. 1. An example of a piecewise-affine function $\phi(\cdot)$ concave in $\mathbb{R}^+$.

$$\phi(y) = \begin{cases} k_0y & \text{if } y \in [0, b_1) \\ k_1y + c_1 & \text{if } y \in [b_1, b_2) \\ \vdots & \\ k_Ny + c_N & \text{if } y \in [b_N, \infty) \end{cases}, \quad \forall y \geq 0, \tag{2}$$

where the scalars k_i, $i = 0, \dots, N$, b_i, $i = 1, \dots, N$, and c_i, $i = 1, \dots, N$, satisfy

$$0 < b_1 < b_2 < \cdots < b_N$$
$$k_0 > k_1 > k_2 > \cdots > k_N$$
$$c_i = \begin{cases} (k_0 - k_1)b_1 & \text{if } i = 1 \\ c_{i-1} + (k_{i-1} - k_i)b_i & \text{if } 2 \leq i \leq N. \end{cases}$$

See Fig. 1 for an example of piecewise-affine concave function in $\mathbb{R}^+$ ($N = 3$).

Note that the results presented in this paper can be also applied to systems of the form

$$x_{k+1} = \hat{A}x_k - \hat{B}\hat{\phi}(y_k),$$

where $\hat{\phi}(\cdot)$ is an odd piecewise-affine function convex in $\mathbb{R}^+$ (it suffices to define $\phi(\cdot) = -\hat{\phi}(\cdot)$, $A = \hat{A}$ and $B = -\hat{B}$).

3. Analysis of the non-linear function

In this section some properties of function $\phi(\cdot)$ are presented. For that purpose the following definition is introduced.

Definition 1. Given the piecewise-affine odd function concave in $\mathbb{R}^+$

$$\phi(y) = \begin{cases} k_0y & \text{if } y \in [0, b_1) \\ k_1y + c_1 & \text{if } y \in [b_1, b_2) \\ \vdots & \\ k_Ny + c_N & \text{if } y \in [b_N, \infty) \end{cases}, \quad \forall y \geq 0,$$

the odd functions $\phi_i(y)$, $i = 1, \dots, N$, are defined as

$$\phi_i(y) = \begin{cases} k_0y & \text{if } y \in [0, d_i) \\ k_iy + c_i & \text{if } y \in [d_i, \infty) \end{cases}, \quad \forall y \geq 0, \tag{3}$$

where $d_i = \frac{c_i}{k_0 - k_i}$, $i = 1, \dots, N$.

Fig. 2 shows functions $\phi_i(\cdot)$, $i = 1, \dots, 3$, corresponding to function $\phi(\cdot)$ of Fig. 1.

It can be observed in Fig. 2 that $\phi(y)$ is the pointwise minimum of $\phi_1(y)$, $\phi_2(y)$ and $\phi_3(y)$. The following lemma states a useful relationship between function $\phi(\cdot)$ and functions $\phi_i(\cdot)$, $i = 1, \dots, N$.

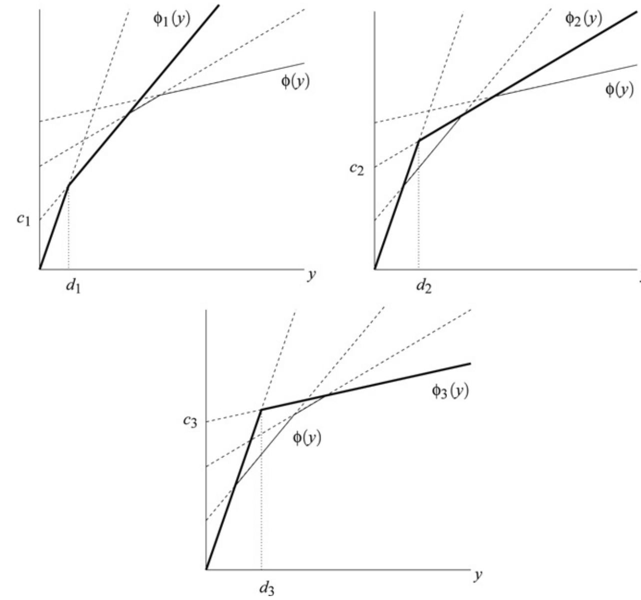

Fig. 2. Functions $\phi_i(\cdot)$, $i = 1, \dots, 3$, corresponding to function $\phi(\cdot)$ of Fig. 1.

Lemma 1 (Hu et al., 2004; Hu & Lin, 2004). *Suppose that $\phi(\cdot)$ is an odd piecewise-affine function concave in $\mathbb{R}^+$. Then*

- $\phi_i(y) \in co\,\{k_0y, \phi(y)\}$, $\forall y \in \mathbb{R}$, $i = 1, \dots, N$.
- $\phi(y) \in co\,\{\phi_1(y), \phi_2(y), \dots, \phi_N(y)\}$, $\forall y \in \mathbb{R}$,

where co stands for convex hull.

4. The *LNL*-invariance notion

In this section, the concept of *LNL*-invariance is presented. This new notion of invariance is stronger than the classical one. However, the *LNL*-invariance has some geometrical properties that allow one to obtain the greatest *LNL*-invariant set by means of a simple algorithm. Moreover, it will be shown that every contractive set for the non-linear system is contained into the greatest *LNL*-invariant set provided by the proposed algorithm.

Definition 2. Consider the system $x_{k+1} = Ax_k - B\phi(Fx_k)$ and let the function $\phi(\cdot)$ be defined as in Eq. (2). Functions $f(x)$ and $f_L(x)$ are defined as

$$\begin{aligned} f(x) &= Ax - B\phi(Fx), \\ f_L(x) &= Ax - Bk_0Fx. \end{aligned} \tag{4}$$

The notion of *LNL*-invariance is introduced in the following definition.

Definition 3. A set Ω is said to be *LNL*-invariant for system $x_{k+1} = Ax_k - B\phi(Fx_k)$ if $x \in \Omega$ implies

$$f(x) = Ax - B\phi(Fx) \in \Omega,$$
$$f_L(x) = Ax - Bk_0Fx \in \Omega.$$

This concept is stronger than simple invariance; that is, if Ω is *LNL*-invariant it is also invariant, but not vice versa.

Remark 1. *LNL* stands for *Linear* and *Non-Linear*. Note that the new constraint $f_L(x) \in \Omega$ added to the concept of *LNL*-invariance is not a very strong constraint as there is a neighborhood of the origin where $f(x)$ equals $f_L(x)$.

Definition 4. We say that $S_0, S_1, \dots, S_k$ is an admissible sequence if $S_i \in \{1, -1\}$, $i = 0, \dots, k$.