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The extended J-spectral factorization for descriptor systems ☆

Delin Chu*, Roger C.E. Tan

Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543, Singapore

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Abstract

In this paper we study an extended J-spectral factorization problem, i.e., J-spectral factorization problem for descriptor systems which may not have full column normal rank and may exhibit poles and zeros on the extended imaginary axis. We present necessary and sufficient solvability conditions and develop a numerically reliable method for the underlying problem. Our method is implemented by using only orthogonal transformations and has an acceptable computational complexity.

Keywords: Spectral factorization; Descriptor system; Orthogonal transformation

1. Introduction

Throughout this paper, $J \in \mathbf{R}^{p\times p}$ and $J' \in \mathbf{R}^{k\times k}$ are two given symmetric matrices; $\mathbf{C}_-$, $\mathbf{C}_0$ and $\mathbf{C}_{0e}$ denote open left complex half plane, imaginary axis and extended imaginary axis, respectively; the square pencil $-sE + A$ is stable if it is regular (i.e., $\det(-sE+A) \neq 0$ for some $s \in \mathbf{C}$) and all its finite generalized eigenvalues are on $\mathbf{C}_-$; $\mathcal{R}^{p\times k}(s)$ and $\mathcal{RL}_\infty^{p\times k}(s)$ denote the sets of $p \times k$ rational matrices and proper rational matrices without poles on $\mathbf{C}_0$, respectively; $G(s) = \left[\begin{array}{c|c} -sE+A & B \\ \hline C & D \end{array}\right]$ means $G(s) = D + C(sE - A)^{-1}B$, and integer k is called its normal rank if $\max_{s\in\mathbf{C}} \operatorname{rank} \begin{bmatrix} -sE+A & B \\ C & D \end{bmatrix} = n + k$, here $E, A \in \mathbf{R}^{n\times n}$.

It is well-known that a large number of quite different types of factorizations of linear systems have been studied in the literature of system theory (Aliev & Larin, 1997; Anderson, 1967; Ball & Ran, 1987; Bart, Gohberg, & Kaashoek, 1986; Chen, 2000; Chen & Francis, 1989; Chen, Lin, & Shamash, 2004; Chu & Ho, 2005; Clements, 1993; Clements, Anderson, Laub, & Matson, 1997; Clements & Glover, 1989; Green, Glover, Limebeer, & Doyle, 1990; Green & Limebeer, 1995; Greenberg, van der Mee, & Protopopescu, 1987; Hara & Sugie, 1991; He & Chen, 2002; Katayama, 1996; Kawamoto, 1999; Kawamoto & Katayama, 2003; Kwakernaak & Sebek, 1994; Oara & Varga, 2000; Wangham & Mita, 1997; Xin & Kimura, 1994; Youla, 1961). Among them, the J-spectral factorization problem has played an important role in optimal Hankel-norm model reduction and H_∞ optimization (Green & Limebeer, 1995; Ball & Ran, 1987), transport theory (Greenberg et al., 1987), and stochastic filtering (Lindquist & Picci, 1991).

Consider the descriptor system of the form

$$E\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{1}$$

where $E, A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{p\times n}$, $D \in \mathbf{R}^{p\times m}$, and the pencil $-sE + A$ is regular.

Definition 1 (*Kawamoto, 1999; Kawamoto & Katayama, 2003*). Given system (1) with all poles in $\mathbf{C}_- \cup \mathbf{C}_{0e}$. The J-spectral factorization problem for the descriptor system (1) is solvable if $G(s) = \left[\begin{array}{c|c} -sE+A & B \\ \hline C & D \end{array}\right]$ has a J-spectral factorization, i.e., there exists an invertible $\Xi(s) \in \mathcal{R}^{m\times m}(s)$ such that:

(i) $G^{\mathrm{T}}(-s)JG(s) = \Xi^{\mathrm{T}}(-s)J'\Xi(s)$.

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* Corresponding author.
E-mail addresses: matchudl@nus.edu.sg (D. Chu), scitance@nus.edu.sg (R.C.E. Tan).

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(ii) All poles and zeros of $\Xi(s)$ lie in $\mathbf{C}_- \cup \mathbf{C}_{0e}$.
(iii) $G(s)\Xi^{-1}(s) \in \mathcal{RL}_\infty^{p\times m}(s)$.

When $E = I_n$ and system (1) has no poles and zeros on $\mathbf{C}_{0e}$, the J-spectral factorization of system (1) reduces to the classical spectral factorization (Bart et al., 1986; Anderson, 1967; Youla, 1961). It is understood (Oara & Varga, 2000) that the difficulties in numerically solving the classical spectral factorization problem are related to the existence of poles and zeros on $\mathbf{C}_{0e}$ and to the noninvertibility of $G^{\mathrm{T}}(-s)JG(s)$. If none of these elements are present, the computation of the classical spectral factorization problem reduces to solving a standard algebraic Riccati equation (Green et al., 1990), as stated in the following theorem.

Theorem 2 (*Green et al., 1990*). *Given a linear time-invariant system*

$$\dot{x} = Ax + Bu, \quad y = Cx + Du. \tag{2}$$

Assume that A is stable, D is of full column rank and the pencil $\begin{bmatrix} -sI+A & B \\ C & D \end{bmatrix}$ has no finite eigenvalues on $\mathbf{C}_0$. Then the J-spectral factorization problem for system (2) is solvable if and only if there exists a nonsingular $D_0 \in \mathbf{R}^{m\times m}$ such that $D^{\mathrm{T}}JD = D_0^{\mathrm{T}}J'D_0$ and the algebraic Riccati equation

$$A^{\mathrm{T}}P + PA + C^{\mathrm{T}}JC - (PB + C^{\mathrm{T}}JD)(D^{\mathrm{T}}JD)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}JC) = 0 \tag{3}$$

has a solution P such that $A - B(D^{\mathrm{T}}JD)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}JC)$ is stable. Moreover, under these two conditions, a J-spectral factor $\Xi(s)$ is given by

$$\Xi(s) = D_0 \left[\begin{array}{c|c} -sI+A & B \\ \hline (D^TJD)^{-1}(D^TJC + B^TP) & I \end{array}\right]. \tag{4}$$

Recently, the J-spectral factorization problem for descriptor system (1) with poles/zeros on $\mathbf{C}_{0e}$ has been considered in Kawamoto (1999) and Kawamoto and Katayama (2003) by using the generalized algebraic Riccati equation approach and the zero compensator technique (Copeland & Safonov, 1992).

Theorem 3 (*Kawamoto, 1999; Kawamoto & Katayama, 2003*). *Given the descriptor system* (1). *Assume that*:

(A1) *All the finite generalized eigenvalues of the pencil $-sE + A$ lie in $\mathbf{C}_- \cup \mathbf{C}_{0e}$*;

(A2) *(E, A, B) is finite dynamic stabilizable and impulse controllable, i.e.*,

$$\mathrm{rank}[-sE + A \;\; B] = n, \quad \forall s \in \mathbf{C}\backslash\mathbf{C}_-,$$

$$\mathrm{rank}[E \;\; A\mathcal{N}(E) \;\; B] = n,$$

where $\mathcal{N}(E)$ denotes a column orthogonal matrix whose columns span the null space of E. Then the J-spectral factorization problem for the descriptor system (1) *is solvable if and only if the pencil $\begin{bmatrix} -sE+A & B \\ C & D \end{bmatrix}$ is of full column rank for some $s \in \mathbf{C}$ and the generalized algebraic Riccati equation*

$$\begin{cases} A_a^{\mathrm{T}}X_a + X_a^{\mathrm{T}}A_a + Q_a - X_a^{\mathrm{T}}\begin{bmatrix} 0 & 0 \\ 0 & (J')^{-1} \end{bmatrix}X_a = 0, \\ E_a^{\mathrm{T}}X_a = X_a^{\mathrm{T}}E_a \end{cases} \tag{5}$$

has a semi-stabilizing solution X_a (for the related definition, we refer to (Kawamoto, 1999; Kawamoto & Katayama, 2003)), here

$$E_a = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \quad A_a = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix},$$

$$Q_a = \begin{bmatrix} C^{\mathrm{T}}JC & C^{\mathrm{T}}JD \\ D^{\mathrm{T}}JC & D^{\mathrm{T}}JD - J' \end{bmatrix}, \quad X_a = \begin{array}{c} \begin{matrix} n & m \end{matrix} \\ \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{matrix} \}n \\ \}m \end{matrix} \end{array}.$$

Furthermore, in this case, a J-spectral factor $\Xi(s)$ is given by

$$\Xi(s) = \left[\begin{array}{c|c} -sE+A & B \\ \hline -(J')^{-1}X_{21} & I-(J')^{-1}X_{22} \end{array}\right]. \tag{6}$$

It is easy to know that the pencil $\begin{bmatrix} -sE+A & B \\ C & D \end{bmatrix}$ is of full column normal rank is a necessary condition for the J-spectral factorization problem of the descriptor system (1). So, Definition 1 excludes system (1) without full column normal rank. The condition (i) in Definition 1 implies that $G^{\mathrm{T}}(-s)JG(s)$ is nonsingular and has a constant inertia that is the same as the inertia of matrix J' for any $s \in \mathbf{C}_0$ that is not a pole of system (1). Obviously, if the descriptor system (1) is not of full column normal rank, then $G^{\mathrm{T}}(-s)JG(s)$ is always singular and its inertia always contains some 0 for any $s \in \mathbf{C}_0$ that is not a pole of system (1). Therefore, in order to include all descriptor systems without full column normal rank in the J-spectral factorization problem, Definition 1 has to be modified. This is the first motivation of the present work.

Assume that $G^{\mathrm{T}}(-s)JG(s)$ has a constant inertia (μ, τ, ν) for almost $s \in \mathbf{C}_0$, then a natural extension of the condition (i) in Definition 1 is that

$$G^{\mathrm{T}}(-s)JG(s) = \Xi^{\mathrm{T}}(-s)\begin{bmatrix} I_\mu & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I_\nu \end{bmatrix}\Xi(s),$$

equivalently,

$$G^{\mathrm{T}}(-s)JG(s) = \hat{\Xi}^{\mathrm{T}}(-s)J'\hat{\Xi}(s),$$

$$\hat{\Xi}(s) = \begin{bmatrix} I_\mu & 0 & 0 \\ 0 & 0 & I_\nu \end{bmatrix}\Xi(s), \quad J' = \begin{bmatrix} I_\mu & 0 \\ 0 & -I_\nu \end{bmatrix},$$

where $\hat{\Xi}(s)$ is of full row rank. This consideration leads us to generalize Definition 1.

Definition 4. $\Xi^{(+)}(s) \in \mathcal{R}^{m\times k}(s)$ is called the Moore–Penrose inverse of $\Xi(s) \in \mathcal{R}^{k\times m}(s)$ if

$$\Xi(s)\Xi^{(+)}(s)\Xi(s) = \Xi(s), \quad \Xi^{(+)}(s)\Xi(s)\Xi^{(+)}(s) = \Xi^{(+)}(s),$$

$$\Xi(s)\Xi^{(+)}(s) = (\Xi(s)\Xi^{(+)}(s))^{\sim}, \quad \Xi^{(+)}(s)\Xi(s) = (\Xi^{(+)}(s)\Xi(s))^{\sim}.$$