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Brief paper

# Inversion of sampled-data system approximates the continuous-time counterpart in a noncausal framework☆

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**Abstract**

Although correspondence between the poles of a continuous-time and sampled-data system with a piecewise constant input is simple and desirable from the stability viewpoint, the relationship between zeros is intricate. Inversion of a sampled-data system is mostly unstable irrespective of the stability of the continuous-time counterpart. This makes it difficult to apply inversion-based control techniques such as perfect tracking, transient response shaping or iterative learning control to sampled-data systems. Although recently developed noncausal inversion techniques help us to circumvent unboundedness of the inversion caused by unstable zeros, whether the inversion of sampled-data systems approximates the continuous-time counterpart or not as the sample period is shortened is still to be determined. This article gives a positive conclusion to this problem.


*Keywords:* Sampled-data systems; Inverse system; Transmission zeros; Discrete-event dynamic system; Feedforward control; Nonminimum phase systems; Zero-order hold; Learning control

## 1. Introduction

In recent times, control systems have typically been installed in digital devices that include samplers and zero-order holds, which convert continuous-time signals into discrete-time signals and vice versa. A zero-order hold generates piecewise constant functions that can approximate any uniformly continuous function $u(t)$ defined on the infinite time horizon, i.e. $\|u(\lfloor t/\tau\rfloor\tau) - u(t)\|_\infty \to 0$ as the sample period $\tau \to 0$ where $\|x(t)\|_\infty = \sup\{|x(t)|;\, t \in (-\infty, +\infty)\}$ and $\lfloor t/\tau\rfloor$ denotes the maximum integer that does not exceed $t/\tau$. This implies that $\| \int_{-\infty}^{+\infty} g(t-\sigma)u(\lfloor\sigma/\tau\rfloor\tau)\,\mathrm{d}\sigma - \int_{-\infty}^{+\infty} g(t-\sigma)u(\sigma)\,\mathrm{d}\sigma\|_\infty \to 0$ for any stable linear system $g$, i.e. the output of stable continuous-time systems with a piecewise constant input $u(\lfloor t/\tau\rfloor\tau)$ approximates the output of the same systems with a continuous input $u(t)$ as the sample period is shortened. This fact encourages us to replace analog controllers with digital controllers with a sufficiently small sample time. In contrast to the earlier mentioned convenient properties, it is recognized that there is no simple correspondence between inversion of the system with continuous input and piecewise constant input. This fact is highlighted by investigations from the viewpoint of transfer functions. Consider a linear causal system with an impulse response $g(t)$. Then, the transfer function is $G(s) = \mathscr{L}[\int_{-\infty}^{+\infty} g(t-\sigma)u(\sigma)\,\mathrm{d}\sigma]/\mathscr{L}[u(t)]$ where $\mathscr{L}$ denotes the one-sided Laplace transform. Assume that the transfer function is expressed as

$$G(s) = \frac{K(s-\gamma_1)(s-\gamma_2)\cdots(s-\gamma_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \tag{1}$$

or $G(s) = c(sI - A)^{-1}b$, where $(A, b, c)$ is a state space representation. Then, the discrete-time transfer function of the system with piecewise constant inputs $u(\lfloor\sigma/\tau\rfloor\tau)$ on the sample time $t = k\tau$ $(k = 0, \pm 1, \ldots)$ is $H_\tau(z) = \mathscr{Z}[\int_{-\infty}^{+\infty} g(k\tau - \sigma)u(\lfloor\sigma/\tau\rfloor\tau)\,\mathrm{d}\sigma]/\mathscr{Z}[u(k\tau)]$; equivalently, $H_\tau(z) = \mathscr{Z}[\mathscr{L}^{-1}[G(s)\mathscr{L}[u(\lfloor t/\tau\rfloor\tau)]](k\tau)]/\mathscr{Z}[u(k\tau)]$, which

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Table 1
Zeros of Euler–Frobenius polynomial $B_{n-m}(z)$

| $n-m$ | Zeros |
|---|---|
| 2 | $-1$ |
| 3 | $-2-\sqrt{3},\ 1/(-2-\sqrt{3})$ |
| 4 | $-5-2\sqrt{6},\ -1,\ 1/(-5-2\sqrt{6})$ |
| 5 | $\lambda_{51},\ \lambda_{52},\ 1/\lambda_{52},\ 1/\lambda_{51}\ (\lambda_{51}\approx -23,\ \lambda_{52}\approx -2.3)$ |
| 6 | $\lambda_{61},\ \lambda_{62},\ -1,\ 1/\lambda_{62},\ 1/\lambda_{61}\ (\lambda_{61}\approx -51,\ \lambda_{62}\approx -4.5)$ |
| $\vdots$ | $\vdots$ |
| odd | $\lambda_{i-11},\ldots,\lambda_{i-1(i-2)/2},\ 1/\lambda_{i-1(i-2)/2},\ldots,1/\lambda_{i-11}$ |
| $i-1$ | $(\lambda_{i-11}<\cdots<\lambda_{i-1(i-2)/2}<-1)$ |
| even | $\lambda_{i1},\ldots,\lambda_{i(i-2)/2},-1,\ 1/\lambda_{i(i-2)/2},\ldots,1/\lambda_{i1}$ |
| $i$ | $(\lambda_{i1}<\cdots<\lambda_{i(i-2)/2}<-1)$ |

is expressed as

$$H_\tau(z)=\frac{cb_\tau\{z-q_1(\tau)\}\cdots\{z-q_{n-1}(\tau)\}}{\{z-\exp(p_1\tau)\}\cdots\{z-\exp(p_n\tau)\}} \tag{2}$$

or $H_\tau(z)=c(zI-A_\tau)^{-1}b_\tau$, where $\mathscr{Z}$ is the one-sided $z$-transform, $A_\tau=\exp(A\tau)$ and $b_\tau=\int_0^\tau \exp(At)b\,\mathrm{d}t$.

Although correspondence between the poles of $G(s)$ and $H_\tau(z)$ is simple and desirable from the stability viewpoint, the relationship between zeros is intricate. It is known that the zeros of $H_\tau(z)$ have the following asymptotic properties in terms of the sample period $\tau$ (Åström, Hagander, & Sternby, 1984; Hagiwara & Araki, 1993): $q_i(\tau)=1+\gamma_i\tau+O(\tau^2)(i=1,\ldots,m)$ and $q_i(\tau)\to$ zeros of $B_{n-m}(z)$ $(i=m+1,\ldots,n-1)$ as $\tau\to 0$, where $B_{n-m}(z)$ is the Euler–Frobenius polynomial, the zeros of which are located on the negative real axis symmetrically with respect to $-1$ (Table 1) (Dubeau & Savoie, 1995; Weller, Moran, Ninness, & Pollington, 2001). This implies that inversion of the discrete-time system $1/H_\tau(z)$ with a small sample period is mostly unstable even if the continuous-time counterpart $1/G(s)$ is stable.

On the other hand, system inversion plays crucial roles in many control applications such as perfect tracking, transient response shaping, disturbance attenuation, and noise cancellation. The aforementioned fact makes it difficult to apply inversion-based control techniques developed for continuous-time systems to sampled-data systems with piecewise constant inputs. For example, consider shaping a transient response of $G(s)$. Then, as long as the zeros of $G(s)$ are stable, one can employ $M(s)/G(s)$ as a prefilter of $G(s)$, where $M(s)$ is a model that has a desired response. However, it is not necessarily possible to apply a corresponding approach directly to the case of piecewise constant inputs because the discrete-time prefilter $N_\tau(z)/H_\tau(z)$ is mostly unstable; here, $N_\tau(z)$ is the discrete-time counterpart of $M(s)$. Nonetheless, one can avoid unboundedness of the prefilter by introducing a discrete-time version of the so-called stable inversion technique, which is a method to apply anticausal convolution to antistable parts of the inverse system and generate bounded outputs (Devasia, Chen, & Paden, 1996; Hunt & Meyer, 1997). Still, even though one circumvents the unboundedness due to unstable zeros of $H_\tau(z)$, whether the discrete-time prefilter $N_\tau(z)/H_\tau(z)$ can be substituted for the continuous-time prefilter $M(s)/G(s)$ is still a question. Recall that $H_\tau(z)$ approximates $G(s)$ for uniformly continuous functions $u(t)$. In such a case, whether $1/H_\tau(z)$ approximates $1/G(s)$ or not must be determined. In this article, the author presents an affirmative conclusion to this problem.

This article is organized as follows: Section 2 defines noncausal stable inversion with the two-sided Laplace transform and $z$-transform and formulates the main problem with illustrative numerical examples; Section 3 demonstrates the main results on an approximation in the noncausal framework; and Section 4 concludes the work.

## 2. Noncausal inversion and formulation of the approximation problem

Since feedback control is essentially causal, the one-sided Laplace transforms and the one-sided $z$-transforms have been widely used as mathematical tools to analyze and design linear feedback controllers. In this framework, transfer functions with unstable poles that are located in the right half plane for continuous-time systems or outside the unit circle for discrete-time systems correspond to diverging signals. This implies that the inverse of systems with unstable zeros is of no practical use. However, feedforward control is not necessarily causal in applications such as perfect tracking, transient response shaping or iterative learning control. Noncausal feedforward control has been proposed to achieve better tracking than that given by causal controllers (Hoover, Longchamp, & Rosenthal, 2004; Hunt & Meyer, 1997; Kinoshita, Sogo, & Adachi, 2002; Kojima & Ishijima, 2003). It is known that noncausality enlarges the application scope of iterative learning control (Markusson, Hjalmarsson, & Norrlöf, 2001; Sun & Wang, 2001; Sun, Wang, & Wang, 2004). In this work, the author introduces the two-sided Laplace transform and the $z$-transform as mathematical tools to analyze noncausal inversion.

The two-sided Laplace transform of a function $f(t)$ where $t\in(-\infty,+\infty)$ is defined as $\mathscr{L}[f(t)](s)=F(s)=\int_{-\infty}^{+\infty}\mathrm{e}^{-st}f(t)\,\mathrm{d}t$, which is an analytic function of $s\in\mathbb{C}$ in the vertical strip area $\gamma_1<\mathrm{Re}(s)<\gamma_2$ (Poularikas, 2000). Let $\alpha$ be a real number satisfying $\gamma_1<\alpha<\gamma_2$. The inverse Laplace transform is then expressed by

$$\begin{aligned}&\mathscr{L}^{-1}[F(s)](t)\\ &\quad=f(t)=\frac{1}{2\pi j}\int_{\alpha-\mathrm{j}\infty}^{\alpha+\mathrm{j}\infty}\mathrm{e}^{st}F(s)\,\mathrm{d}s\\ &\quad=\begin{cases}\sum_{\mathrm{Re}(p_n)<\alpha}\mathrm{Res}(\mathrm{e}^{st}F(s),\,p_n), & t\geqslant 0,\\ \sum_{\mathrm{Re}(p_m)>\alpha}\mathrm{Res}(-\mathrm{e}^{st}F(s),\,p_m), & t<0,\end{cases}\end{aligned} \tag{3}$$

where $\{p_n\}$ and $\{p_m\}$ are the sets of poles of $F(s)$. For example, consider a bounded function $f(t)$ defined as $f(t)=\mathrm{e}^{-t}$ for $t\geqslant 0$ and $f(t)=\mathrm{e}^{2t}$ for $t<0$. Then, we have $F(s)=1/(s+1)+1/(2-s)$, which is analytic on $\{s;\,-1<\mathrm{Re}(s)<2\}$.

The two-sided $z$-transform of a discrete-time function $h(k)$ where $k\in\mathbb{Z}$ is defined as $\mathscr{Z}[h(k)](z)=H(z)=\sum_{k=-\infty}^{+\infty}h(k)z^{-k}$, which is an analytic function of $z\in\mathbb{C}$ in the annular domain $r_0<|z|<R_0$. Let $\alpha$ be a positive real