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LTI approximation of nonlinear systems via signal distribution theory[☆]

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Abstract

L_2 and L_1 optimal linear time-invariant (LTI) approximation of discrete-time nonlinear systems, such as nonlinear finite impulse response (NFIR) systems, is studied via a signal distribution theory motivated approach. The use of a signal distribution theoretic framework facilitates the formulation and analysis of many system modelling problems, including system identification problems. Specifically, a very explicit solution to the L_2 (least squares) LTI approximation problem for NFIR systems is obtained in this manner. Furthermore, the L_1 (least absolute deviations) LTI approximation problem for NFIR systems is essentially reduced to a linear programming problem. Active LTI modelling emphasizes model quality based on the intended use of the models in linear controller design. Robust stability and LTI approximation concepts are studied here in a nonlinear systems context. Numerical examples are given illustrating the performance of the least squares (LS) method and the least absolute deviations (LAD) method with LTI models against nonlinear unmodelled dynamics.

Keywords: Approximation; Identification; Least absolute deviation; Least squares; Near-linearity; Nonlinear system

1. Introduction

This paper provides several applications of signal distribution theory to active linear modelling of nonlinear systems (with the terminology active modelling, in contrast to passive modelling, it is emphasized that we have in mind an intended use of the linear models: here in linear controller design for nonlinear systems). Linear time-invariant (LTI) models are widely used in control applications although real plants are typically at least mildly nonlinear. In the present work we solve, among other things, an L_2 LTI approximation problem for single-input, single-output (SISO), discrete-time, nonlinear finite impulse response (NFIR) systems. Furthermore, we solve also an analogous L_1 LTI approximation problem for NFIR systems.

The theory of uniform distribution of sequences (Kuipers and Niederreiter, 1974) has its origin in some important work of Weyl (Körner, 1988; Kuipers and Niederreiter, 1974). This theory provides a realistic nonstochastic signal analysis framework. It has been used in the analysis of pseudorandom number generators and in numerical integration (Knuth, 1998; Niederreiter, 1992), to mention two important application areas, see also Grabner et al. (1999). The generalized harmonic analysis (GHA) of Wiener (1927, 1930) is another realistic nonstochastic signal analysis framework (Ljung, 2001; Mäkilä et al., 1998). However, the scope of GHA as a signal analysis tool is much more limited than the theory of distributions. In Mäkilä (2003), Mäkilä and Partington (2004) distribution concepts are used in the analysis of nonlinear approximation of systems.

The problem of optimal LTI modelling of nonlinear systems has received considerable attention in recent years, see for example Sastry (1999), Ljung (2001) and Mäkilä and Partington (2003). One reason for this interest is the desire to understand the impact of nonlinear distortions on the performance of standard LTI model estimation methods (Enqvist and Ljung, 2002, 2003; Ljung, 2001; Pintelon and Schoukens, 2001; Schoukens et al., 2001). Furthermore, it is important to understand the design of LTI controllers based on LTI approximations of nonlinear systems, see for example Zames (1966), Sastry (1999) and Mäkilä and Partington (2003).

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Our solution to the L_2 LTI approximation problem for NFIR systems is based on earlier work in Mäkilä (2004), Mäkilä and Partington (2004). The use of signal distribution theory allows us to provide a very explicit solution to the approximation problem for an important class of inputs. The L_1 LTI approximation problem for NFIR systems is here shown to be equivalent to certain classical continuous and discrete L_1 approximation problems (Pinkus, 1989; Powell, 1981). Note that the L_1 LTI approximation problem is not studied in Mäkilä (2004), Mäkilä and Partington (2004). Our results are related to asymptotic analysis of LTI model identification via least squares (LS) and least absolute deviations (LAD) methods for NFIR systems. For the LS approach with LTI models such analysis is presented for NFIR systems in the very interesting thesis Enqvist (2003) using stochastic techniques, see also Enqvist and Ljung (2005).

Our aim is to obtain results that allow the comparison of LS and LAD techniques. The LS case allows a more detailed analysis and thus this case is our main object of study here. Connections to correlation and LS techniques for identification of linear models are also established. We are especially interested in understanding how LTI modelling techniques react to unmodelled nonlinear dynamics when the LTI models are used in LTI controller design. For the LS case some pertinent results are also described in Mäkilä (2004, 2005). The L_2 LTI approximation problem is analysed for a class of nonlinear state space (NSS) systems using for the first time a sample limit result in Mäkilä and Partington (2004).

To understand the performance of LTI models in LTI controller design for nonlinear systems, it is essential to analyse when LTI controllers can perform well in a nonlinear context. The concept of nearly linear system is interesting in this respect (Mäkilä, 2005). This concept demonstrates that in the nonlinear case, it is useful to consider a generalization of the concept of best LTI model of a nonlinear system (Mäkilä and Partington, 2003). We discuss this generalization in some detail, via the new concept of nonlinear companion, and provide associated robust stability analysis. This analysis indicates that it would be desirable to generalize robust control theory to unmodelled dynamics which is not of the standard sector form, see also Mäkilä (2006). Similarly, it would be desirable to generalize performance analysis of system identification and model validation to include unmodelled dynamics which is not of the sector form.

The rest of the paper is organized as follows. Section 2 introduces some notation and concepts. An LS LTI approximation problem for NFIR systems is solved in Section 3 via a signal distribution approach allowing us to interpret the LS problem as an L_2 approximation problem. An LAD LTI approximation problem is studied in Section 4 by writing it as a certain L_1 approximation problem. A generalized L_2 approximation problem is solved in Section 5. Furthermore, some connections to LTI system identification are established. An L_2 approximation problem for a class of NSS systems is studied in Section 6. The generalization of the best approximation setup in Mäkilä and Partington (2003) is considered in Section 7. This is motivated also via robust stability considerations when the intended use of the linear model is in controller design. An illustrative example is given in Section 8. Some conclusions are drawn in Section 9.

2. Preliminaries

Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ denote the nonnegative integers, the integers, and the reals, respectively. Let $\mathrm{card}(V) \in \mathbb{N}$ denote the number of elements in a finite set V. The space $s(\mathbb{N})$ is the linear space of all real sequences $x=\{x(k)\in\mathbb{R}\}_{k\in\mathbb{N}}$ over $\mathbb{N}$. The linear normed space $\ell_\infty(\mathbb{N})$ is the space of all real sequences $x=\{x(k)\in\mathbb{R}\}_{k\in\mathbb{N}}$ such that $\|x\|_\infty \equiv \sup_{k\in\mathbb{N}} |x(k)| < \infty$. The linear normed space $\ell_1(\mathbb{N})$ denotes the space of all real sequences $x=\{x(k)\in\mathbb{R}\}_{k\in\mathbb{N}}$ such that $\|x\|_1 \equiv \sum_{k\in\mathbb{N}} |x(k)| < \infty$.

Let $x \in s(\mathbb{N})$ and introduce the autocovariance of the signal x as

$$R_{xx}(k) \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{t=0}^{n-1} x(t)x(t+k), \quad k \in \mathbb{Z}.$$

(Here we put $x(t)=0$ for $t<0$.) Clearly $R_{xx}(-k)=R_{xx}(k)$ whenever either $R_{xx}(-k)$ or $R_{xx}(k)$ exists. The autocovariance sequence $R_{xx}=\{R_{xx}(k)\in\mathbb{R}\}_{k\in\mathbb{Z}}$ need not exist for a general signal x. We say that a signal $x \in s(\mathbb{N})$ possessing an autocovariance sequence R_{xx} allows GHA (Wiener, 1927, 1930), or simply that x is then a *GHA signal*.

A signal x is said to be *quasistationary* (Ljung, 2001) if $x \in \ell_\infty(\mathbb{N})$ and x possesses an autocovariance sequence R_{xx}. Furthermore, we say that the two real sequences v and x possess a crosscovariance sequence if

$$R_{vx}(k) \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{t=0}^{n-1} v(t)x(t+k)$$

exists for any $k \in \mathbb{Z}$. We write $R_{vx}=\{R_{vx}(k)\in\mathbb{R}\}_{k\in\mathbb{Z}}$. We shall denote $R_{vx}^+=\{R_{vx}(k)\in\mathbb{R}\}_{k\in\mathbb{N}}$.

3. An L_2 approximation problem

In this section an L_2 LTI approximation problem for strictly causal NFIR systems will be solved.

3.1. A preliminary analysis

Let $G : D(G; s(\mathbb{N})) \to s(\mathbb{N})$ denote a strictly causal NFIR system given as

$$y(t) = (Gu)(t) = g(u(t-1), \ldots, u(t-m)), \tag{1}$$

where y is the output, $u \in D(G; s(\mathbb{N}))$ is the input, $g : D(g; \mathbb{R}^m) \to \mathbb{R}$ is a mapping and m is a positive integer. Here $D(G; s(\mathbb{N}))$ denotes the domain of definition of G in $s(\mathbb{N})$, i.e. the set of all $u \in s(\mathbb{N})$ such that $y = Gu \in s(\mathbb{N})$. $D(G; s(\mathbb{N}))$ is determined by the domain $D(g; \mathbb{R}^m)$ of the mapping g.

An L_2 LTI approximation problem for G is now introduced as follows. Let $u \in D(G; s(\mathbb{N}))$. Consider

$$\inf_{\Gamma} \lim_{n\to\infty} \frac{1}{n} \sum_{t=0}^{n-1} [(Gu)(t) - (\Gamma u)(t)]^2, \tag{2}$$