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Brief paper

An observer for non-linear differential-algebraic systems☆

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Abstract

In this paper, we consider design of observers for non-linear models containing both dynamic and algebraic equations, so-called differential-algebraic equations (DAE), of index 1. The observer is formulated as a DAE that, by construction, has index 1. The main results of the paper include conditions that ensure local stability of the observer error dynamics. Design methodology is presented and illustrated using a small simulation study.

Keywords: Non-linear observer; Differential-algebraic equations; Descriptor system

1. Introduction and observer formulation

State observation for non-linear ODE models has been studied for quite some time and is still an active area of research. The focus of this work is to design observers for models containing both dynamic and algebraic equations, so called differential-algebraic equations (DAE) or descriptor models, of index 1. State observation for linear DAEs has been studied by, for example, Nikoukhah, Willsky, and Levy (1992) using the Kalman filter. Non-linear DAEs are considered in e.g. Becerra, Roberts, and Griffiths (2001) where an extension of the extended Kalman filter is used and also by Zimmer and Meier (1997), where the original DAE model is rewritten as an ODE on a restricted manifold (Rheinboldt, 1984). Other works include (Boutayeb & Darouach, 1995) that uses linearization techniques and (Kidane, Yamashita, & Nishitani, 2003) that, in addition to a linearization procedure, employs index reduction techniques to cope with high-index models. In Kaprielian and Turi (1992), a Lyapunov-based approach is used in the design of the observer.

Now, our approach is introduced and the observer structure is presented. Usually, an observer is formulated as an ODE, $\dot{\hat{x}} = k(\hat{x}, u, y)$, for some vector field k. However, this work is an extension of a work by Nikoukhah (1995, 1998) where it is noted that the requirement that the observer must be formulated as an ODE can be relaxed to a class of index 1 DAEs. This is due to the fact that low index DAEs are no more difficult to integrate than ODEs (Hairer & Wanner, 1996; Brenan, Campbell, & Petzold, 1996). First, we briefly outline the idea proposed in Nikoukhah (1998) for ODE models, and then our observer formulation for DAE models is presented.

Consider the state-space model given by

$$\dot{x} = f(x, u),$$

$$y = h(x),$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^m$ the measurement vector and $u \in \mathbb{R}^k$ the known control input. An often used observer is then

$$\dot{\hat{x}} = f(\hat{x}, u) + g(\lambda),$$

$$0 = y - h(\hat{x}) + \lambda,$$

using a, perhaps not so common, formulation using a slack variable λ. The function $g(\lambda)$ is the observer feedback used to ensure stability of the error dynamics. In Nikoukhah (1998), a similar formulation, using slack variables, is used to define a

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more general class of observers in the form

$$\dot{\hat{x}} = f(\hat{x}, u) + h_x(\hat{x})^{\mathrm{T}}\dot{\lambda} + G(\hat{x}, u)\lambda, \tag{1a}$$

$$0 = y - h(\hat{x}). \tag{1b}$$

This observer is, under some mild technical assumptions, shown to be a DAE of index 1. The observer has some connections to reduced order observers but does not inherit the possibly poor noise properties of reduced order observers. A discussion on this and other properties of the observer can be found in Nikoukhah (1995, 1998).

Here, a similar approach is adopted for designing state estimators for the following class of semi-explicit models:

$$\dot{x}_1 = f(x_1, x_2, z, t), \tag{2a}$$

$$0 = h(x_1, x_2, z, t), \tag{2b}$$

where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ are the unknown system variables and $z \in \mathbb{R}^{n_z}$ the vector of known signals, and $h \in \mathbb{R}^m$. The vector z includes both measurements and control signals and possibly other known quantities. Eq. (2b) can include both measurement equations and algebraic constraints. Exact conditions on f and h are given in Section 3 and it is assumed that the model (2) has index 1. For a definition of index, see e.g. Hairer and Wanner (1996); Brenan et al. (1996).

The observer formulation used, here, for estimating x_i in (2a), based on the known z, is

$$\dot{\hat{x}}_1 = f(\hat{x}_1, \hat{x}_2, z, t) + F(t)\dot{\lambda} + G(t)\lambda, \tag{3a}$$

$$0 = h(\hat{x}_1, \hat{x}_2, z, t), \tag{3b}$$

where $\lambda \in \mathbb{R}^r$ and $r = m - n_2$. The observer gains F and G are the available design variables, which have to be chosen such that the observer has index 1 and provides a convergent state estimate.

The outline of the paper is as follows. First, Section 2 shows how to ensure that the observer has index 1 such that the numerical integration of the observer is generally possible. Secondly, local stability of the estimator error dynamics is explored in Section 3. The design method is summarized and exemplified in Section 4 where a simulation example, based on components in an air suspension system of a heavy duty truck, is considered.

2. Observer index

The objective of this section is to give conditions on the observer gain F such that the observer (3) is a DAE with index 1. Before we can do that, some auxiliary subspaces of $\mathbb{R}^{n_1}$ have to be introduced. First, define the space

$$\mathscr{V} = \{x_1 :\ (x_1, x_2)^{\mathrm{T}} \in N(h_x) \text{ for some } x_2\}. \tag{4}$$

This means that $\mathscr{V}$ is the truncation of the null space $N(h_x)$ to $\mathbb{R}^{n_1}$, where h_x denotes the partial derivative of h with respect to x. The first lemma shows that the dimension of the space is preserved under this truncation.

Lemma 1. *If h_x has full row rank and h_{x_2} has full column rank, then* $\dim \mathscr{V} = \dim N(h_x)$.

Proof. Since h_x has full row rank and $r = m - n_2$, we have

$$N(h_x) = \operatorname{span}\{x^1, \ldots, x^{n_1-r}\}, \tag{5}$$

where $\{x^1, \ldots, x^{n_1-r}\}$ is a linearly independent set. Using the notation

$$x^i = \begin{pmatrix} x_1^i \\ x_2^i \end{pmatrix}$$

it follows from the definition that $\mathscr{V} = \operatorname{span}\{x_1^1, \ldots, x_1^{n_1-r}\}$. We have to prove that $x_1^1, \ldots, x_1^{n_1-r}$ are linearly independent, so assume, therefore, that

$$\sum_i \mu^i x_1^i = 0. \tag{6}$$

It follows from (5) that $\sum_i \mu^i x^i \in N(h_x)$ and consequently

$$h_{x_1}\left(\sum_i \mu^i x_1^i\right) + h_{x_2}\left(\sum_i \mu^i x_2^i\right) = 0.$$

Using that h_{x_2} has full column rank and assumption (6) we obtain $\sum_i \mu^i x_2^i = 0$. Together with assumption (6), this implies that

$$\sum_i \mu^i x^i = 0.$$

It follows that $\mu^1 = \cdots = \mu^{n_1-r} = 0$, since $x^1, \ldots, x^{n_1-r}$ are linearly independent. Hence $x_1^1, \ldots, x_1^{n_1-r}$ are linearly independent as well, which proves the lemma. □

Let $\mathscr{W}$ be an algebraic complement of $\mathscr{V}$ in $\mathbb{R}^{n_1}$, i.e. $\mathscr{W}$ is a subspace such that each $u \in \mathbb{R}^{n_1}$ has a unique representation $u = v + w$, where $v \in \mathscr{V}$ and $w \in \mathscr{W}$. Let P_v and P_w denote the associated projections defined by $v = P_v u$ and $w = P_w u$. Now, we can state and prove the main result of this section.

Theorem 1. *Suppose that h_x has full row rank and that h_{x_2} has full column rank. It then follows that* $\dim \mathscr{W} = r$ *and if $F(t) \in \mathbb{R}^{n_1 \times r}$ is chosen so that*

$$\operatorname{Im} F(t) = \mathscr{W} \tag{7}$$

then observer (2) *has index* 1.

Proof. It follows from Lemma 1 that $\dim \mathscr{W} = r$ and that $F(t)$ has full column rank. By taking (3a) and differentiating (3b) with respect to t we get in matrix notation

$$\begin{bmatrix} I & 0 & -F \\ h_{x_1} & h_{x_2} & 0 \end{bmatrix} \begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} f + G\lambda \\ -h_z\dot{z} - h_t \end{pmatrix}. \tag{8}$$

That the observer has index 1 is equivalent to that the matrix on the left-hand side is invertible. It is therefore sufficient to show that the homogeneous problem

$$x_1 - F\lambda = 0, \tag{9a}$$

$$h_{x_1}x_1 + h_{x_2}x_2 = 0 \tag{9b}$$