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Brief paper

Synchronizing linear systems via partial-state coupling ☆

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Abstract

A basic result in the synchronization of linear systems via output coupling is presented. For identical discrete-time linear systems that are detectable from their outputs and neutrally stable, it is shown that a linear output feedback law exists under which the coupled systems globally asymptotically synchronize for all fixed connected (asymmetrical) network topologies. An algorithm is provided to compute such a feedback law based on individual system parameters.

Keywords: Multi-agent system; Synchronization; Consensus algorithm; Partial-state coupling; Output feedback

1. Introduction

A notable meeting point for many researchers from different fields is the topic *synchronization*. One of the reasons for that comes from nature as synchronization in large networks of dynamical systems is a frequently encountered phenomenon in biology. Among many others, one can count synchronously discharging neurons, crickets chirping in accord, and metabolic synchrony in yeast cell suspensions. Another reason is the abundance of technological applications: coupled synchronized lasers, vehicle formations, and sensor networks, just to name a few. We refer the reader to the surveys Boccaletti, Latora, Moreno, Chavez, and Hwang (2006), Olfati-Saber, Fax, and Murray (2007), Strogatz (2001) and Wang (2002) for references and more examples.

The main issue in studying the synchronization of coupled dynamical systems is the stability of synchronization. As in all cases where stability is the issue, the question whose answer is sought is *Under what conditions* will the individual systems synchronize? In a simplified yet widely-studied scenario, where the individual system dynamics are identical and the coupling between them is linear, studies focus on two ingredients: the dynamics of an individual system and the network topology. Starting with the *agreement algorithm* in Tsitsiklis, Bertsekas, and Athans (1986) a number of contributions (Angeli & Bliman, 2006; Blondel, Hendrickx, Olshevsky, & Tsitsiklis, 2005; Jadbabaie, Lin, & Morse, 2003; Moreau, 2005; Olfati-Saber & Murray, 2004; Ren & Beard, 2005) have gathered around the case where the weakest possible assumptions are made on the network topology at the expense of restrictive individual system dynamics. It was established in those works on *multi-agent systems* that when the individual system is taken to be an integrator and the coupling is of full-state, synchronization (*consensus*) results for time-varying interconnections whose unions[1] over an interval are assumed to be connected instead of that each interconnection at every instant is connected.

Another school of research investigates networks with more complicated (nonlinear) individual system dynamics. When that is the case, the restrictions on the network topology have to be made stricter in order to ensure stability of synchronization. Generally speaking, more than mere connectedness of the network has been needed: coupling strength is required to be

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[1] By *union of interconnections* we actually mean the union of the graphs representing the interconnections.

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larger than some threshold and sometimes a symmetry[2] or balancedness assumption is made on the connection graph. Different (though related) approaches have provided different insights over the years. The primary of such approaches is based on the calculations of the eigenvalues of the connection matrix and a parameter (e.g. the maximal Lyapunov exponent) depending on the individual system dynamics (Pecora & Carroll, 1998; Wu & Chua, 1995). In endeavor to better understand synchronization stability, tools from systems theory such as Lyapunov functions (Belykh, Belykh, & Hasler, 2006; Hui, Haddad, & Bhat, 2007), passivity (Arcak, 2007; Stan & Sepulchre, 2007), contraction theory (Slotine, Wang, & El-Rifai, 2004), and incremental input to state stability (δISS) theory (Cai & Chen, 2006) have also proved useful.

This paper studies a broad class of linear systems under weak assumptions on the coupling structure and generalizes some of the existing results on synchronization. Namely, we consider identical individual discrete-time linear systems interacting via (diffusive) output coupling under a fixed (time-invariant) network topology. The contribution of the paper is in proving (via construction) the following basic result, which seems to have been missing from the literature. For a linear system[3] that is neutrally stable and detectable from its output, there always exists a linear output feedback law that ensures the global asymptotic synchronization of any connected (not necessarily symmetric nor balanced) network of any number of coupled replicas of that system. To fortify our contribution practically, we provide an algorithm to compute one such feedback law. It is worth noting that our main theorem makes a compromise result between the two previously mentioned cases (i) where synchronization is established for very primitive individual system dynamics, such as that of an integrator, but under the weakest conditions on the network topology and (ii) where the network topology has to satisfy stronger conditions, such as that the coupling strength should be above a threshold, for want of achieving synchronization for nonlinear individual system dynamics.

The remainder of the paper is organized as follows. Notation and definitions reside in the next section. We give the problem statement along with our assumptions in Section 3. In Section 4 we provide a preliminary synchronization result on a network of linear systems with orthogonal system matrices. Then we generalize that result to establish our main theorem in Section 5.

2. Notation and definitions

The number of elements in a (finite) set $\mathcal{S}$ is denoted by $\#\mathcal{S}$. Let $\mathbb{N}$ denote the set of nonnegative integers. Let $|\cdot|$ denote 2-norm. Identity matrix in $\mathbb{R}^{n\times n}$ is denoted by I_n. A matrix $Q \in \mathbb{R}^{n\times n}$ is *orthogonal* if $QQ^{\mathrm{T}} = Q^{\mathrm{T}}Q = I_n$. Orthogonal matrices satisfy $|Qv| = |v|$ for all $v \in \mathbb{R}^n$. Given $C \in \mathbb{R}^{m\times n}$ and $A \in \mathbb{R}^{n\times n}$, pair (C, A) is *observable* if $[C^{\mathrm{T}}\ A^{\mathrm{T}}C^{\mathrm{T}}\ A^{2T}C^{\mathrm{T}}\ \ldots\ A^{(n-1)T}C^{\mathrm{T}}]$ is full row rank. Pair (C, A) is *detectable* (in the discrete-time sense) if that $CA^kx = 0$ for some $x \in \mathbb{R}^n$ and for all $k \in \mathbb{N}$ implies $\lim_{k\to\infty} A^kx = 0$. Matrix $A \in \mathbb{R}^{n\times n}$ is *neutrally stable* (in the discrete-time sense) if it has no eigenvalue with magnitude greater than unity and the Jordan block corresponding to an eigenvalue λ with $|\lambda| = 1$ is of size one.[4] Let $\mathbf{1} \in \mathbb{R}^p$ denote the vector with all entries equal to unity.

Kronecker product of $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is

$$A \otimes B := \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

In the pages to come we will enjoy the properties $(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$ (provided that products AC and BD are allowed), $A\otimes B + A\otimes C = A\otimes(B + C)$ (for B and C that are of same size) and $|A\otimes B| = |A||B|$.

Matrix $P \in \mathbb{R}^{n\times n}$ is an *orthogonal projection* onto the subspace range(P) if $P^2 = P$ and $P^{\mathrm{T}} = P$. For an orthogonal projection P, if the columns of $C^{\mathrm{T}} \in \mathbb{R}^{n\times m}$ are an orthonormal basis for range(P) then $P = C^{\mathrm{T}}C$. Matrix $V = I_n - P$ is also an orthogonal projection and range(V) = range$(P)^{\perp}$. It is easy to see that $PV = VP = 0$.

A (*directed*) *graph* is a pair $(\mathcal{N}, \mathcal{A})$ where $\mathcal{N}$ is a nonempty finite set (of *nodes*) and $\mathcal{A}$ is a finite collection of (ordered) pairs (*arcs*) (n_i, n_j) with $n_i, n_j \in \mathcal{N}$. A *path* from n_1 to n_ℓ is a sequence of nodes $\{n_1, \ldots, n_\ell\}$ such that (n_i, n_{i+1}) is an arc for $i \in \{1, \ldots, \ell - 1\}$. A graph is *connected* if it has a node to which there exists a path from every other node.[5]

The graph of a matrix $\Lambda := [\lambda_{ij}] \in \mathbb{R}^{p\times p}$ is the pair $(\mathcal{N}, \mathcal{A})$ where $\mathcal{N} = \{n_1, \ldots, n_p\}$ and $(n_i, n_j) \in \mathcal{A}$ iff $\lambda_{ij} > 0$. Matrix Λ is said to be *connected* (in the discrete-time sense) if it satisfies:

(i) $\lambda_{ii} > 0$ and $\lambda_{ij} \geq 0$ for all i, j;
(ii) each row sum equals 1;
(iii) its graph is connected.[6]

For Λ that is connected, it is known that $\lim_{k\to\infty} \Lambda^k = \mathbf{1}r^{\mathrm{T}}$ where $r \in \mathbb{R}^p$ has nonnegative entries and satisfies $r^{\mathrm{T}}\mathbf{1} = 1$. We mention that, in an interconnection of systems, if the matrix describing the network topology satisfies properties (i) and (ii) above, then the coupling between the systems is said to be *diffusive*.

Given maps $\xi_i : \mathbb{N} \to \mathbb{R}^n$ for $i \in \{1, \ldots, p\}$ and a map $\bar\xi : \mathbb{N} \to \mathbb{R}^n$, the elements of the set $\{\xi_i(\cdot) : i = 1, \ldots, p\}$ are said to *synchronize to* $\bar\xi(\cdot)$ if $|\xi_i(k) - \bar\xi(k)| \to 0$ as $k \to \infty$ for all i.

[2] A network is *symmetric* if the matrix representing it is symmetric.
[3] $x^+ = Ax + u;\ y = Cx$.

[4] Note that A is neutrally stable iff there exists a symmetric positive definite matrix P such that $A^{\mathrm{T}}PA - P \leq 0$ (Antsaklis & Michel, 1997).
[5] Note that this definition of connectedness for directed graphs is weaker than strong connectivity and stronger than weak connectivity.
[6] Recall that for continuous-time applications, definition of connectedness is different: a matrix $[\gamma_{ij}]$ is considered connected (in the continuous-time sense) if $\gamma_{ij} \geq 0$ for $i \neq j$; each row sum equals 0; and its graph is connected.