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Synthesis of optimal controllers for piecewise affine systems with sampled-data switching☆

Shun-ichi Azuma[a,*], Jun-ichi Imura[b]

[a]*Department of Systems Science, Graduate School of Informatics, Kyoto University, Uji, Kyoto 611-0011, Japan*
[b]*Department of Mechanical and Environmental Informatics, Graduate School of Information Science and Engineering, Tokyo Institute of Technology, 2-12-1 O-okayama, Meguro-ku, Tokyo 152-8552, Japan*

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Abstract

This paper discusses the optimal control problem of the continuous-time piecewise affine (PWA) systems with *sampled-data switching*, where the switching action is executed based upon a condition on the state at each sampling time. First, an algebraic characterization for the problem to be feasible is derived. Next, an optimal continuous-time controller is derived for a general class of PWA systems with sampled-data switching, for which the optimal control problem is feasible but whose subsystems in some modes may be *uncontrollable* in the usual sense. Finally, as an application of the proposed approach, the high-speed and energy-saving control problem of the CPU processing is formulated, and the validity of the proposed methods is shown by numerical simulations.

Keywords: Hybrid systems; Piecewise affine systems; Sampled-data; Switching; Optimal control; Feasibility

1. Introduction

The optimal control problem of hybrid systems is one of the fundamental and challenging research topics, and various approaches have been obtained so far. Bemporad and Morari (1999) and Bemporad, Borrelli, and Morari (2000) have proposed an optimal control approach based on the mixed logical dynamical (MLD) system model and Sakizlis, Dua, Perkins, and Pistikopoulos (2002) have approximately solved the continuous-time optimal control problem of the continuous-time MLD systems. Sussmann (1999), Riedinger and Iung (1999), Riedinger, Kratz, Iung, and Zanne (1999), and Shaikh and Caines (2003), respectively, have studied the optimal control problem of hybrid systems from the viewpoint of the maximum principle. On the other hand, Branicky, Borkar, and Mitter (1998), Branicky, Johansen, Petersen, and Frazzoli (2000), Rantzer and Johansson (2000), Hedlund and Rantzer (2002), Gokbayrak and Cassandras (2000), and Xu and Antsaklis (2000), respectively, have developed the dynamic programming method for the optimal control of hybrid systems. Furthermore, it has been solved by using various techniques such as the constraint programming (Bemporad & Giorgetti, 2003), the graph search algorithm (Stursberg, 2004a,b), the reachability specification (Lygeros, Tomlin, & Sastry, 1999), and the state feedback technique (Bemporad, Giua, & Seatzu, 2002).

On the other hand, in many industrial hybrid systems such as cars with an automatic transmission and production facilities in chemical process, the transition of the discrete state (or simply, the discrete transition) occurs according to a switching decision by the digital device, i.e., depending upon whether a condition on the continuous/discrete state holds or not at each sampling time fixed in the digital device. Thus such switching is called here the *sampled-data switching*. It is remarked that in hybrid systems with the sampled-data switching, the discrete transition (if possible) takes place only at each sampling time, while the continuous state behavior on a sampling time interval

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* Corresponding author. Tel.: +81 774 38 3952; fax: +81 774 38 3945.
E-mail addresses: sazuma@i.kyoto-u.ac.jp (S.-i. Azuma), imura@cyb.mei.titech.ac.jp (J.-i. Imura).

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is the same as the behavior of the so-called continuous-time linear/affine system. However, for the optimal control problem of such hybrid systems, few practical results have been obtained. In fact, the standard continuous-time models (e.g., the continuous-time PWA/MLD/LC system models), in which the discrete transitions occur depending upon the *current* state, cannot capture the solution behaviors of hybrid systems with the sampled-data switching, since the discrete transition by the sampled-data switching, in general, depends on the past state as well as the current state due to holding the value of the discrete state on each sampling time interval. In addition, the discrete-time model based approaches (e.g., Bemporad et al., 2000; Bemporad & Morari, 1999) will not necessarily be practical for such hybrid systems, because the control performance on the continuous-time domain may be undesirable if the sampling period is not sufficiently small, and on the contrary, the computation amount for solving the optimal control problem exponentially grows as the sampling period is taken smaller under the condition that the control time period is fixed. Note also that the controllability property of a controlled plant with the sampled-data switching is not necessarily preserved by its discrete-time model, in other words, the discrete-time model for some controllable PWA system with sampled-data switching is uncontrollable. This is due to the differences of the class of the control input signals to be considered in the real plant and the model, that is, the class of the control input signals in the discrete-time model is limited to a class of piecewise *constant* functions even when the piecewise *continuous* functions can be used for a control input in a controlled plant (Imura, 2004).

Thus for the class of hybrid systems with sampled-data switching, Silva and Krogh (2001), and Imura (2004) have introduced new continuous-time models called the sampled-data hybrid automaton model and the sampled-data PWA system model, respectively, where the sampled-data switching is explicitly expressed and the piecewise continuous functions are considered as the control input. By explicitly using these sampled-data models, the following advantages can be obtained: (i) the control performance on the continuous-time domain can be taken into consideration, (ii) the controllability property of controlled plants is preserved. Motivated by the above facts, in Imura (2004), some preliminary results for solving the optimal control problem in the simple case in which the subsystem in every mode is controllable in the usual sense have been presented as the first step to develop the synthesis of optimal controllers for the sampled-data PWA systems. However, in Imura (2004), the feasibility problem has not been addressed and the optimal controllers may be insufficient in the sense that the class of the systems to which the controllers can be applied is limited.

This paper thus addresses the optimal control problem including the feasibility problem for a general class of sampled-data PWA systems. First, a necessary and sufficient condition for the optimal control problem to be feasible is derived, which characterizes the feasible state space for checking the feasibility of the problem, and further as its complementary result for practical use, an *easily checkable* sufficient condition based on a graph technique is derived. It will turn out from this feasibility analysis that the optimal control problem of sampled-data PWA systems may be often feasible even if it includes some uncontrollable subsystems. Next, based on the preliminary results in Imura (2004), an optimal continuous-time control design method for a general class of sampled-data PWA systems, i.e., the systems whose subsystems are *not* all controllable in the usual sense, is proposed. Finally, as an interesting example among many applications to this kind of systems, the high-speed and energy-saving control problem of the CPU processing is considered. More specifically, we first formulate the model of the CPU system composed of a CPU, a buffer, and a cooling fan, and then its control problem is considered as the optimal control problem for the sampled-data PWA system model. Next, its feasibility is verified by the sufficient condition derived here, and it is shown by numerical simulations that the proposed controller is effective.

Notation: let $\mathbf{R}$, $\mathbf{N}$, $\mathbf{N}_+$, and $\mathbf{PC}$ denote the real number field, the set of nonnegative integers, the set of positive integers, and the set of all piecewise continuous functions, respectively. We denote by $0_{n\times m}$, $\mathtt{I}_n$ and $\mathtt{1}_n$ (or simply, 0, $\mathtt{I}$, and $\mathtt{1}$) the $n \times m$ zero matrix, the $n\times n$ identity matrix, and the $n\times 1$ vector whose all elements are 1, respectively. The symbol $x_{(i)}$ denotes the ith element of the vector x, and the vector inequality $x_1 \leqslant (<) x_2$ denotes that each element of $x_1 - x_2$ is nonpositive (negative). We use $\mathrm{span}(V)$ to express the space spanned by the column vectors of the matrix V, and we use $\lambda_i(M)$ $(i = 1, 2, \ldots, n)$ to express the eigenvalues of the $n \times n$ matrix M. The cardinality of the finite set $\mathbf{I}$ is denoted by $\mathrm{card}(\mathbf{I})$, and the set $\mathbf{S}$ given as the form $\mathbf{S} := \{x \in \mathbf{R}^n | Ax + b \leqslant 0, Cx + d < 0\}$ is called here the *polyhedron* (or more precisely, the *convex polyhedron*), where A, C and b, d are some matrices and column vectors, respectively. Finally, $\mathrm{vert}(\mathbf{P})$ expresses the set of all vertices of the closed polyhedron $\mathbf{P} := \{x \in \mathbf{R}^n | Ax + b \leqslant 0\}$.

2. Sampled-data PWA systems and problem formulation

2.1. Sampled-data PWA systems

This paper focuses on the class of hybrid systems with sampled-data switching in Fig. 1, which involves, in addition to PWA dynamics, the digital device that consists of the sampler, the logic, and the holder. To express solution behaviors of such a system in a rigorous way, the following model Σ_{sd}, called the *sampled-data PWA system model* (Imura, 2004), is introduced:

$$\Sigma_{\mathrm{sd}} : \begin{cases} \dot{x}(t) = A_{I(t)}x(t) + B_{I(t)}u(t) + a_{I(t)}, \\ I(t) = I(t_k), \quad \forall t \in [t_k, t_{k+1}), \\ x(t_{k+1}) = \phi(h, I(t_k), x(t_k), u_{[t_k,t_{k+1}]}), \\ I(t_{k+1}) = I_+ \quad \text{if} \quad x(t_{k+1}) \in \mathbf{S}_{I_+}, \end{cases} \tag{1}$$

where $x \in \mathbf{R}^n$ is the continuous state, $I \in \mathbf{I}$ is the discrete state (it is sometimes called the *mode*), $\mathbf{I} := \{0, 1, \ldots, M-1\}$ is the set on which the discrete state takes values, $M \in \mathbf{N}_+$ is the number of the discrete state values, $u \in \mathbf{R}^m$ is the control input, $A_I \in \mathbf{R}^{n\times n}$, $B_I \in \mathbf{R}^{n\times m}$, and $a_I \in \mathbf{R}^n$ are constant matrices for mode I, $h \in \mathbf{R}$ is the switching decision time period, $t_k \in \mathbf{R}$ is the time defined as $t_k := kh$ for $k \in \mathbf{N}$, which is called here the *switching decision time*, and $I_+ \in \mathbf{I}$ is