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ScienceDirect
Automatica 44 (2008) 504–511
automatica
www.elsevier.com/locate/automatica

Brief paper

Uniqueness conditions for the affine open-loop linear quadratic differential game☆

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Received 11 January 2006; received in revised form 9 March 2007; accepted 8 June 2007
Available online 14 September 2007

Abstract

In this note we consider the open-loop Nash linear quadratic differential game with an infinite-planning horizon. The performance function is assumed to be indefinite and the underlying system affine. We derive both necessary and sufficient conditions under which this game has a unique Nash equilibrium.

Keywords: Linear-quadratic games; Open-loop Nash equilibrium; Affine systems; Solvability conditions; Riccati equations

1. Introduction

In the last decades, there is an increased interest in studying diverse problems in economics and optimal control theory using dynamic games (Engwerda, 2005b). In particular in environmental economics and macroeconomic policy coordination, dynamic games are a natural framework to model policy coordination problems (see e.g. the books and references in Dockner, Jørgensen, van Long, & Sorger, 2000; Engwerda, 2005b; Plasmans, Engwerda, van Aarle, Di Bartolomeo, & Michalak, 2006). In these problems, the open-loop Nash strategy is often used as one of the benchmarks to evaluate outcomes of the game. In optimal control theory it is well-known that, e.g. the issue to obtain robust control strategies can be approached as a dynamic game problem (see e.g. Başar & Bernhard, 1995).

In this note we consider the open-loop linear quadratic differential game. This problem has been considered by many authors and dates back to the seminal work of Starr and Ho (1969) (see, e.g. Abou-Kandil, Freiling, & Jank, 1993; Başar & Olsder, 1999; Eisele, 1982; Engwerda, 1998a, 1998b; Feucht, 1994; Haurie & Leitmann, 1984; Kremer, 2002; Lukes & Russell, 1971; Meyer, 1976; Weeren, 1995). More specifically, we study in this paper the (regular indefinite) infinite-planning horizon case. The corresponding regular definite (that is the case that the state weighting matrices Q_i (see below) are semi-positive definite) problem has been studied, e.g. extensively in Engwerda (1998a, 1998b). Kremer (2002) (see also Kremer & Stefan, 2002) studied the regular indefinite case using a functional analysis approach, under the assumption that the uncontrolled system is stable. In particular, these papers show that, in general, the infinite-planning horizon problem does not have a unique equilibrium. Moreover Kremer (2002) shows that whenever the game has more than one equilibrium, there will exist an infinite number of equilibria. Furthermore the existence of a unique solution is related to the existence of a so-called LRS solution of the set of coupled algebraic Riccati equations, see (4). Unfortunately these results obtained for stable systems cannot be directly used to derive results for stabilizable systems using a feedback transformation. This, since such a transformation in general corrupts the open-loop information structure of the problem (see e.g. Engwerda, 2005c where this point is illustrated).

In Engwerda (2005a) (see also Engwerda, 2005b) the above results were generalized for stabilizable systems, using a state-space approach, for a performance criterion that is a pure quadratic form of the state and control variables. In this note we

☆ This paper was presented at the CDC-ECC'05 conference in Sevilla, Spain (Engwerda, 2005d). This paper was recommended for publication in revised form by Associate Editor Masayuki Fujita under the direction of Editor Ian Petersen.

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0005-1098/$ - see front matter
doi:10.1016/j.automatica.2007.06.006

generalize this result for performance criteria that also include "cross-terms", i.e. products of the state and control variables. Performance criteria of this type often naturally appear in economic policy making and have been studied, e.g. in Engwerda, van Aarle, and Plasmans (1999) and Kremer (2002). In this paper we, moreover, assume that the linear system describing the dynamics is affected by a deterministic variable. For a finite-planning horizon the corresponding open-loop linear quadratic game has been studied in Başar and Olsder (1999).

The outline of this note is as follows. Section 2 introduces the problem and contains some preliminary results. The main results of this paper are stated in Section 3, whereas Section 4 contains some concluding remarks. The proofs of the main theorems are included in the Appendix.

2. Preliminaries

In this paper we assume that player $i=1,2$ likes to minimize:

$$\lim_{t_f\to\infty} J_i(t_f,x_0,u_1,u_2) \quad \text{where } J_i(t_f,x_0,u_1,u_2)$$

$$:= \int_0^{t_f} [x^{\mathrm T}(t),\ u_1^{\mathrm T}(t),\ u_2^{\mathrm T}(t)]M_i \begin{bmatrix} x(t)\\ u_1(t)\\ u_2(t)\end{bmatrix} \mathrm{d}t, \qquad (1)$$

$$M_i = \begin{bmatrix} Q_i & V_i & W_i\\ V_i^{\mathrm T} & R_{1i} & N_i\\ W_i^{\mathrm T} & N_i^{\mathrm T} & R_{2i}\end{bmatrix}, \quad R_{ii}>0,\ i=1,2,$$

and $x(t)$ is the solution from the linear differential equation

$$\dot x(t) = Ax(t) + B_1u_1(t) + B_2u_2(t) + c(t), \quad x(0)=x_0. \qquad (2)$$

The variable $c(.)\in L_2$ here is some given trajectory. Notice that we make no definiteness assumptions w.r.t. matrix Q_i.

We assume that the matrix pairs (A,B_i), $i=1,2$, are stabilizable. So, in principle, each player is capable to stabilize the system on his own.

The open-loop information structure of the game means that both players only know the initial state of the system and that the set of admissible control actions are functions of time, where time runs from zero to infinity. We assume that the players choose control functions belonging to the set of square integrable functions yielding a stable closed-loop system (see also e.g. Trentelman, 1989)

$$\mathcal{U}_s(x_0) = \left\{ u\in L_2(0,\infty) \,\middle|\, \lim_{t_f\to\infty} J_i(t_f,x_0,u) \in \mathbb{R}\cup\{-\infty,\infty\}, \right.$$
$$\left. \lim_{t\to\infty} x(x_0,u,t)=0 \right\}.$$

Here $x(x_0,u,t)$ is the solution of (2).[1] Notice that the assumption that the players use simultaneously stabilizing controls introduces the cooperative meta-objective of both players to stabilize the system (see e.g. Engwerda, 2005b for a discussion). For simplicity of notation we will omit from now on the dependency of $\mathcal{U}_s$ on x_0.

In the rest of the paper the algebraic Riccati equations (see the end of the paper for the introduced notation)

$$A^{\mathrm T}K_i + K_iA - (K_iB_i+V_i)R_{ii}^{-1}(B_i^{\mathrm T}K_i + V_i^{\mathrm T})$$
$$+ Q_i = 0, \quad i=1,2, \qquad (3)$$

and the set of (coupled) algebraic Riccati equations

$$0 = \tilde A_2^{\mathrm T}P + P\tilde A - PBG^{-1}\tilde B^{\mathrm T}P + \tilde Q \qquad (4)$$

or, equivalently,

$$0 = A_2^{\mathrm T}P + PA - \left(PB + \begin{bmatrix} Z_1\\ Z_2\end{bmatrix}\right) G^{-1}(\tilde B^{\mathrm T}P + Z) + Q$$

play a crucial role.

Definition 2.1. A solution $P^{\mathrm T} =: (P_1^{\mathrm T}, P_2^{\mathrm T})$, with $P_i\in\mathbb{R}^n$, of the set of algebraic Riccati equations (4) is called

(a) *stabilizing*, if $\sigma(\tilde A - BG^{-1}\tilde B^{\mathrm T}P)\subset\mathbb{C}^-$;[2]
(b) *left–right stabilizing*[3] (LRS) if
 (i) it is a stabilizing solution, and
 (ii) $\sigma(-\tilde A_2^{\mathrm T} + PBG^{-1}\tilde B^{\mathrm T})\subset\mathbb{C}_0^+$.

The next relationship between certain invariant subspaces of matrix M and solutions of the Riccati equation (4) is well-known (see e.g. Engwerda et al., 1999). This property can also be used to calculate the (left–right) stabilizing solutions of (4).

Lemma 2.2. *Let $V\subset\mathbb{R}^{3n}$ be an n-dimensional invariant subspace of M, and let $X_i\in\mathbb{R}^{n\times n}$, $i=0,1,2$, be three real matrices such that*

$$V = \mathrm{Im}[X_0^{\mathrm T},\ X_1^{\mathrm T},\ X_2^{\mathrm T}]^{\mathrm T}.$$

If X_0 is invertible, then $P_i := X_iX_0^{-1}$, $i=1,2$, solves (4) *and $\sigma(A-BG^{-1}(Z+\tilde B^{\mathrm T}P)) = \sigma(M|_V)$. Furthermore, (P_1,P_2) is independent of the specific choice of basis of V.*

Lemma 2.3.

1. *The set of algebraic Riccati equations* (4) *has an LRS solution (P_1,P_2) if and only if matrix M has an n-dimensional stable graph subspace and M has $2n$ eigenvalues (counting algebraic multiplicities) in $\mathbb{C}_0^+$.*
2. *If the set of algebraic Riccati equations* (4) *has an LRS solution, then it is unique.*

[1] $\lim_{t_f\to\infty} J_i(t_f,x_0,u) = -\infty(\infty)$ if $\forall r\in\mathbb{R}, \exists T_f\in\mathbb{R}$ such that $t_f\geqslant T_f$ implies $J_i(t_f,x_0,u)\leqslant r(\geqslant r)$.

[2] $\sigma(H)$ denotes the spectrum of matrix H; $\mathbb{C}^- = \{\lambda\in\mathbb{C} \mid \mathrm{Re}(\lambda)<0\}$; $\mathbb{C}_0^+ = \{\lambda\in\mathbb{C}\mid \mathrm{Re}(\lambda)\geqslant 0\}$.

[3] In Engwerda (2005b) such a solution is called strongly stabilizing.