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Reachability analysis of constrained switched linear systems☆

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Abstract

In this note, we investigate the reachability of switched linear systems with switching/input constraints. We prove that, under a mild assumption of the feasible switching signals, the reachability set is the reachable subspace of the unconstrained system. We also address the local reachability for switched linear systems with input constraints and present a complete criterion for a general class of switched linear systems.

Keywords: Switched linear systems; Switching signal; Reachability; Switching constraints; Input constraints; Directed graph

1. Introduction

A switched linear system is a hybrid system which consists of several linear time-invariant subsystems and a switching rule that specifies the switching among the subsystems. Switched linear systems have been attracting much attention in recent years and the reader is referred to Liberzon (2003), Sun and Ge (2005) for recent developments.

For controllability and reachability of switched linear systems, much work has been done (Ezzine & Haddad, 1989; Krastanov & Veliov, 2005; Stanford & Conner, 1980; Sun, Ge, & Lee, 2002; Xie & Wang, 2003). In particular, for the unconstrained systems, complete geometric and equivalent algebraic criteria have been presented in Sun et al. (2002), Gurvits (2002), Xie and Wang (2003).

Recently, Krastanov and Veliov (2005) extended the controllability condition to the case where the control input is constrained in a cone. In many practical situations, more constraints are usually imposed to the systems. For example, in workshops, the order of the activated subsystems is pre-assigned rather than arbitrarily assigned. This imposes a restriction on the switching signal. Another example is the control input which is subject to a certain saturation that imposes a restriction on the input signal.

In this note, we focus on the following input/switching constraints: the switching signal is governed by a directed graph, the switching times are in given intervals, and/or the control input is bounded. The contribution of this work includes (i) complete reachability criteria are presented for the constrained switched linear systems under mild assumptions; and (ii) the practical implications of the results are briefly discussed.

2. Preliminaries

For a natural number k, we denote by $\bar{k}$ the set $\{1, \ldots, k\}$.

A switched linear control system is in the form

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the continuous state, $u_t \in \mathbf{R}^p$ is the control input, $\sigma(t) \in \bar{m}$ is the discrete state, also known as the switching signal, $A_i \in \mathbf{R}^{n\times n}$ and $B_i \in \mathbf{R}^{n\times p}$, $i \in \bar{m}$ are real constant matrices.

In this work, we assume that the switching signal is continuous on the right, that is, $\lim_{s\to t+} \sigma(s) = \sigma(t),\ \forall t$. The ordered sequence of the switching times $t_0, t_1, t_2, \ldots$ is said to be the switching time sequence, and the ordered discrete state sequence $\sigma(t_0), \sigma(t_1), \sigma(t_2), \ldots$ is said to be the switching index sequence. The sequence $(t_0, \sigma(t_0)), (t_1, \sigma(t_1)), (t_2, \sigma(t_2)), \ldots$

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is said to be the switching sequence of the switching signal, and will be denoted by SS_σ in short. The right-continuity of the switching signal guarantees that the signal is well-defined, i.e., the number of switches is finite in any finite time interval.

For clarity, let $\sum(A_i, B_i)_{i=1}^m$ or $\sum(A_i, B_i)_{\bar{m}}$ denote the continuous dynamics (1) where both the input and switching are totally unconstrained. Let $\phi(t; t_0, x_0, u, \sigma)$ denote the continuous state at time t of switched system (1) starting from $x(t_0)=x_0$ with input u and switching signal σ.

Let $\mathscr{S}$ be the allowed set of switching signals, and $\mathscr{U}$ be the allowed set of inputs.

Definition 2.1. The reachable set of system (1) at time $T > 0$ starting from x under $\mathscr{S}$ and $\mathscr{U}$, denoted $R(x, T, \mathscr{U}, \mathscr{S})_{\bar{m}}$, is the set $\{\phi(T; 0, x, u, \sigma) : u \in \mathscr{U},\ \sigma \in \mathscr{S}\}$. System (1) is said to be (completely) reachable under $\mathscr{S}$ and $\mathscr{U}$, if there is a time $T > 0$, such that $R(x, T, \mathscr{U}, \mathscr{S})_{\bar{m}} = \mathbf{R}^n$, $\forall x \in \mathbf{R}^n$. System (1) is said to be locally reachable at x_0, if there is a time $T > 0$, such that x_0 is an interior point of the set $R(x_0, T, \mathscr{U}, \mathscr{S})_{\bar{m}}$.

Let $\mathscr{V}(A_i, B_i)_{\bar{m}}$ be the minimum subspace of $\mathbf{R}^n$ which is invariant under all A_i, $i \in \bar{m}$ and contains all the image spaces of B_i, $i \in \bar{m}$.

Lemma 2.1 (Sun et al., 2002). *For the unconstrained switched linear system $\sum(A_i, B_i)_{\bar{m}}$, the reachable set is precisely the subspace $\mathscr{V}(A_i, B_i)_{\bar{m}}$.*

We thus refer to $\mathscr{V}(A_i, B_i)_{\bar{m}}$ as the reachable subspace of system (1).

Lemma 2.2 (Sun et al., 2002). *For any given matrices $A_k \in \mathbf{R}^{n\times n}$ and $B_k \in \mathbf{R}^{n\times p_k}$, $k = 1, 2$, inequality*

$$rank[A_1 e^{A_2 t} B_1, B_2] \geqslant rank[A_1 B_1, B_2] \tag{2}$$

holds for almost all $t \in \mathbf{R}$.

For a time T and a switching signal σ with switching sequence $(t_0, i_0), (t_1, i_1), \ldots, (t_l, i_l)$ in $[0, T)$, define

$$\begin{aligned}\mathscr{D}(\sigma, T) \stackrel{\text{def}}{=} & e^{A_{i_l}(T-t_l)} \cdots e^{A_{i_1}(t_2-t_1)} \langle A_{i_0}|B_{i_0}\rangle + \cdots \\ & + e^{A_{i_l}(T-t_l)} \langle A_{i_{l-1}}|B_{i_{l-1}}\rangle + \langle A_{i_l}|B_{i_l}\rangle,\end{aligned} \tag{3}$$

where $\langle A|B\rangle \stackrel{\text{def}}{=} \mathrm{Im}[B, AB, \ldots, A^{n-1}B]$ is the reachable subspace of the pair (A, B).

3. Main results

In reality, the switching transition is usually governed by a logic-based switching device. Here, we assume that the switching index sequence is governed by a directed graph. Similarly, we assume that the switching time is generated within a given interval.

Suppose that G is a directed graph composed of the set $\bar{m}$ of nodes, and a set of directed arcs N, where $N \subseteq \bar{m} \times \bar{m}$. For each $k \in \bar{m}$, let $N(k) = \{i \in \bar{m}: (k, i) \in N\}$.

Suppose that the switched linear control system is constrained by

$$\begin{aligned} SS_\sigma = & (t_0, i_0), (t_1, i_1), (t_2, i_2), \ldots, \\ & i_k \in N(i_{k-1}), \quad t_k - t_{k-1} \in \Delta, \quad k = 1, 2, \ldots, \end{aligned} \tag{4}$$

where $\Delta = \Delta(t_{k-1}, \sigma(t_{k-1}), x(t_{k-1}))$ is in the form of a time interval (s_1, s_2) with $0 < s_1 < s_2 < \infty$.

In the above model, $G = (\bar{m}, N)$ specifies the allowed switches from one subsystem to others. That is, $N(k)$ defines the allowed subsystem indices following the kth subsystem. In other words, a switching index sequence in the form $\ldots, k, i, \ldots$ with $i \notin N(k)$ is prohibited. Accordingly, if N is a strict subset of $\bar{m} \times \bar{m}$, then the directed graph imposes a nontrivial restriction on the switching index sequence and hence on the switching signal. Similarly, Δ specifies the allowed period of duration for each switch. The duration depends possibly on the continuous/discrete states, which reflects many practical situations.

Let $\mathscr{S}_{G,\Delta}$ be the set of switching signals satisfying (4).

Given any sequence $i_1, \ldots, i_l$, we say the sequence generates the set L if $i_j \in L$ for any $j = 1, \ldots, l$ and each element in L appears at least once in the sequence. For example, the sequence 4, 1, 3, 4, 3, 1 generates the set $\{1, 3, 4\}$.

Recall that a closed path or circuit in a directed graph is an alternating sequence of nodes and arcs

$$v_0, x_1, v_1, \ldots, x_k, v_k$$

in which each arc x_i is $v_{i-1}v_i$ and $v_0 = v_k$. Each path generates a subset of the nodes which contains each point in the path.

The following theorem presents a sufficient condition for complete reachability of the constrained switched linear systems.

Theorem 3.1. *Suppose that L is a subset of $\bar{m}$ and directed graph G permits a closed path which generates set L. If the unconstrained switched system $\Sigma(A_i, B_i)_L$ is completely reachable, then, the constrained switched system* (4) *is also completely reachable.*

Proof. Suppose that the closed path that generates set L is $k_1 \to k_2 \to \cdots \to k_\kappa \to k_1$. Let $\{i_1, \ldots, i_j\}$ be a permutation of L which is a subsequence of $k_1, \ldots, k_\kappa$. That is, there is a natural number sequence $1 \leqslant l_1 < l_2 < \cdots < l_j \leqslant \kappa$, such that $i_v = k_{l_v}$, $v = 1, \ldots, j$.

First, pick up a switching path ϱ with

$$\begin{aligned} SS_\varrho = & \{(t_0, i_1), (t_1, i_2), \ldots, (t_{j-1}, i_j), (t_j, i_1), \\ & (t_{j+1}, i_2), \ldots, (t_{2j-1}, i_j), \ldots\}, \end{aligned} \tag{5}$$

where the switching times $t_1, t_2, \ldots$ are design parameters.

By the proof of Sun et al. (2002, Theorem 1) for almost any strictly monotone time sequence $\{t_1, t_2, \ldots, t_\imath\}$ with $\imath = \sum_{\mu=0}^{n-1} j(jn)^\mu$, we have $\mathscr{D}(\varrho, t_\imath) = \mathscr{V}(A_i, B_i)_L$.

Next, consider a switching path σ with

$$\begin{aligned} SS_\sigma = & \{(s_0, k_1), (s_1, k_2), \ldots, (s_{\kappa-1}, k_\kappa), (s_\kappa, k_1), \\ & (s_{\kappa+1}, k_2), \ldots, (s_{2\kappa-1}, k_\kappa), \ldots\}, \end{aligned} \tag{6}$$

where the switching times $s_1, s_2, \ldots$ are design parameters.