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Brief paper

Averaging of nonsmooth systems using dither☆

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Abstract

It was shown by Zames and Shneydor and later by Mossaheb that a high-frequency dither signal of a quite arbitrary shape can be used to narrow the effective nonlinear sector of Lipschitz continuous feedback systems. In this paper, it is shown that also discontinuous nonlinearities of feedback systems can be narrowed using dither, as long as the amplitude distribution function of the dither is absolutely continuous and has bounded derivative. The averaged system is proven to approximate the dithered system with an error of the order of dither period.

Keywords: Averaging theory; Discontinuous control; Dither; Hybrid systems; Switched systems; Nonsmooth systems

1. Introduction

A frequently used technique to stabilize a nonlinear feedback system in Luré form is by injecting a high-frequency dither signal, which narrows the nonlinear sector. If the dither frequency is sufficiently high, the behavior of the dithered system will be qualitatively the same as an averaged system, whose nonlinearity is the convolution of the amplitude distribution of the dither and the original nonlinearity. Analysis and control design can then be carried out on the averaged system, which in most cases is simpler to analyze due to lack of external dither signal and narrower nonlinearity. For the case when the original nonlinearity is Lipschitz continuous, the scheme outlined above was rigorously justified using properties of the amplitude distribution function of the dither Zames & Shneydor, 1976, 1977). Similar results were obtained later using classical averaging theory (Mossaheb, 1983).

The Lipschitz continuity assumption on the nonlinearity of the dithered system is often violated in practice. Indeed, discontinuous nonlinearities in feedback systems with high-frequency excitations appear in a large variety of applications, including systems with adaptive control (Åström & Wittenmark, 1989), friction (Armstrong-Helouvry, 1991), power electronics (Lehman & Bass, 1996), pulse-width modulation (Peterchev & Sanders, 2001), quantization (Gray & Neuhoff, 1998), relays (Tsypkin, 1984), and variable-structure control (Utkin, 1992). It is common to analyze these systems using empirical methods such as describing functions, which can give a quite good intuitive understanding. It is hard, however, to get bounds on the approximation these methods provide and they may even give erroneous results, so therefore there is a need for a solid treatment of discontinuous systems with high-frequency excitation. Recently, certain classes of these systems have been thoroughly studied, such as power converters (Lehman & Bass, 1996), pulse-width modulated systems (Gelig & Churilov, 1998; Teel, Moreau, & Nesic, 2004), relay systems (Iannelli, Johansson, Jönsson, & Vasca, 2003a), and stick-slip drives (Sedghi, 2003).

The main contribution of the paper is an averaging theorem for a general class of nonsmooth systems with a quite arbitrary

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periodic dither. The result states that the dithered and the averaged systems have qualitatively the same behavior when the dither has sufficiently high frequency and an absolutely continuous amplitude distribution function with bounded derivative. The averaging theorem might be interpreted as an extension to nonsmooth feedback system of previous results, which were limited to Lipschitz-continuous systems (Zames & Shneydor, 1976, 1977; Mossaheb, 1983).

The outline of the paper is as follows. The dithered system and the corresponding averaged system are introduced in Section 2. The amplitude distribution function of the dither signal is thoroughly discussed, since it plays a key role in the analysis. The main result on the approximation error between the dithered and the averaged systems is presented in Section 3. The paper is concluded in Section 4 and the proofs are reported in Appendix.

2. Preliminaries

2.1. Dithered system

The dithered feedback system is defined as

$$\dot{x}(t) = f_0(x(t), t) + \sum_{i=1}^{m} f_i(x(t), t) n_i(g_i(x(t), t) + \delta_i(t)),$$
$$x(0) = x_0. \tag{1}$$

The state x belongs to $\mathbb{R}^q$. The functions $f_i : \mathbb{R}^q \times \mathbb{R} \to \mathbb{R}^q$, $i = 1, \ldots, m$, are assumed to be globally Lipschitz with respect to both x and t, i.e., there exists a positive constant L_f such that for all $x_1, x_2 \in \mathbb{R}^q$ and $t_1, t_2 \geqslant 0$,

$$|f_i(x_1, t_1) - f_i(x_2, t_2)| \leqslant L_f(|x_1 - x_2| + |t_1 - t_2|).$$

We further assume that f_0 is piecewise continuous with respect to t, $f_0(0, t) = 0$ for all $t \geqslant 0$, and

$$|f_0(x_1, t) - f_0(x_2, t)| \leqslant L_f |x_1 - x_2|$$

for all $x_1, x_2 \in \mathbb{R}^q$ and $t \geqslant 0$. Similarly, the functions $g_i : \mathbb{R}^q \times \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, m$, are assumed to have a common Lipschitz constant $L_g > 0$, i.e.,

$$|g_i(x_1, t_1) - g_i(x_2, t_2)| \leqslant L_g(|x_1 - x_2| + |t_1 - t_2|)$$

for all $x_1, x_2 \in \mathbb{R}^q$, $t_1, t_2 \geqslant 0$. The nonlinearities $n_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, m$, are assumed to be functions of bounded variation. Recall that the total variation TV of a function $n : \mathbb{R} \to \mathbb{R}$ is

$$TV(n) \triangleq \sup_{-\infty < z_0 \leqslant z_1 \leqslant \cdots \leqslant z_k < \infty} \sum_{i=1}^{k} |n(z_i) - n(z_{i-1})|,$$

where the supremum is taken over all finite sequences $\{z_i\}_{i=0}^k$ with $k \geqslant 1$ (Wheeden & Zygmund, 1977). If the total variation is bounded, we simply say that n is of bounded variation. Hence, the functions n_i can be discontinuous, but they are necessarily bounded. Each dither signal $\delta_i : [0, \infty) \to \mathbb{R}$ is supposed to be a p-periodic measurable function bounded by a positive constant M_δ, i.e., $|\delta_i| \leqslant M_\delta \ \forall i$.

When the differential equation (1) has a discontinuous right-hand side (due to that at least one n_i is discontinuous), existence and uniqueness of solutions depend critically on the considered definition of solution (Filippov, 1988). In the following, we assume that the differential equation (1) has at least one absolutely continuous solution $x(t, x_0)$ on $[0, \infty)$ (in the sense of Carathéodory). We suppose that the time intervals when the solution is at a discontinuity point of n_i are of zero Lebesgue measure. Note that as a consequence, we do not consider solutions with sliding modes. Furthermore, we suppose that the solutions have no accumulation of switching events (Zeno solutions).

The assumptions on system (1) imply that there exists a positive constant L_x such that $|x(t_1) - x(t_2)| \leqslant L_x |t_1 - t_2|$ for almost all $0 \leqslant t_1 \leqslant t_2 < \infty$. Estimates of the Lipschitz constant L_x can be easily obtained on any compact interval.

Remark 1. The assumption on the nonlinearity n_i is weak. The class of considered systems thus contains quite exotic differential equations for which, for example, existence and uniqueness of solution cannot easily be addressed. However, for most cases in applications the existence of a Carathéodory solution is reasonable. Existence and uniqueness of solutions for dithered relay systems are discussed in Iannelli, Johansson, Jönsson, and Vasca (2004).

Remark 2. The assumption on global Lipschitz continuity of the functions f_i, g_i is used to derive the Lipschitz bound L_x. The assumption can be relaxed by assuming Lipschitzness on a bounded set provided that dithered and averaged solutions belong to such set, see Teel and Nesic (2000).

2.2. Dither signals and their amplitude distribution functions

Definition 2.1. The amplitude distribution function $F_\delta : \mathbb{R} \to [0, 1]$ of a p-periodic dither signal $\delta : [0, \infty) \to \mathbb{R}$ is defined as

$$F_\delta(\xi) \triangleq \frac{1}{p} \mu(\{t \in [0, p) : \delta(t) \leqslant \xi\}),$$

where μ denotes the Lebesgue measure.

When the amplitude distribution function is absolutely continuous (with respect to its Lebesgue measure), the amplitude density function $f_\delta(\xi)$ is defined as

$$f_\delta(\xi) \triangleq \frac{\mathrm{d}F_\delta}{\mathrm{d}\xi}(\xi),$$

which exists almost everywhere.

The amplitude density and amplitude distribution functions play in a deterministic framework the same role as the probability density and cumulative distribution functions play in a stochastic framework. In particular, the amplitude distribution function is bounded, monotonously increasing, continuous from the right, and, if it is absolutely continuous, its derivative corresponds to the amplitude density function.