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Brief paper

Piecewise affinity of min–max MPC with bounded additive uncertainties and a quadratic criterion☆

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Abstract

This brief shows how a min–max MPC with bounded additive uncertainties and a quadratic cost function results in a piecewise affine and continuous control law. Proofs based on properties of the cost function and the optimization problem are given. The boundaries of the regions in which the state space can be partitioned are also treated. The results are illustrated by an example.

Keywords: Min–max MPC algorithm; Predictive control; Min–max optimization; Piecewise linear controllers; Uncertainty

1. Introduction

Model predictive control (MPC) is one of the control techniques able to cope with both model uncertainties and constraints in an explicit way. There are different approaches for modelling uncertainties. The approach considered here is that of bounded additive or global uncertainties (Camacho & Bordóns, 2004); this supposes that all uncertainties can be globalized in a single vector which is added to the 1-step ahead prediction equation. When bounded uncertainties are considered explicitly, it would seem that more robust control would be obtained if the controller minimized the objective function for the worst-case situation.

Min–max MPC (MMMPC) techniques have been used to explicitly consider the effect of the uncertainty on the control law (Campo & Morari, 1987; Casavola, Giannelli, & Mosca, 2000; Veres & Norton, 1993; Lee & Yu, 1997; Kim, Kwon, & Lee, 1998). However, all of these have a great computational burden in common which limits the range of processes to which they can be applied. When the cost function is based on 1 or ∞ norms the min–max problem can be efficiently solved using linear programming techniques (Allwright & Papavasilou, 1992). In other works (Kothare, Balakrishnan, & Morari, 1996; Lu & Arkun, 2000), the computational burden is lessened by minimizing an upper bound of the worst case instead of explicitly solving a min–max problem.

MMMPC controllers can be divided into two types: open-loop and closed-loop min–max predictive controllers. In the first type, predictions are computed in an open-loop manner (although the resulting controller is a feedback controller). These controllers are based on the solution of a single min–max problem optimizing a single control policy for all possible values of the uncertainty. Closed-loop min–max predictive controllers take into account that the control law is actually applied in a feedback manner when computing the predictions. These controllers employ different strategies such as nested min–max problems (Bemporad, Borrelli, & Morari, 2003; Lee & Yu, 1997), optimization of multiple control policies (Kerrigan & Maciejowski, 2004; Scokaert & Mayne, 1998), and, more recently, feasibility constraints (Sakizlis, Kakalis, Dua, & Pistikopoulos, 2004) when minimizing the nominal or expected cost. Open-loop MMMPC is known to be very conservative, whereas closed-loop MMMPC is known to suffer from a much greater computational burden.

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Bemporad et al. (2003) have shown that both open-loop or closed-loop MMMPC with ∞-norm (or 1-norm) have a piecewise affine (PWA) nature. This fact was deduced by the use of multiparametric programming and it allows explicit solutions of such control laws. In this brief we show that the constrained MMMPC control law with a quadratic objective function is also PWA and continuous. We provide proofs based mainly on the properties of the cost function and on the optimization problem. This result can be exploited to implement this type of control law to processes with fast dynamics. The results presented in the paper can be applied to open-loop prediction MMMPC or to MMMPC using a semi-feedback strategy (Mayne, 2001). In this, some kind of feedback is introduced into the predictions because the system is pre-controlled using an inner feedback gain. This technique (Rossiter, Kouvaritakis, & Rice, 1998) is known to reduce the conservatism of open-loop predictive controllers (Bemporad, 1998; Löfberg, 2003) without having to increase the computational burden.

The brief is organized as follows: Section 2 presents the MMMPC strategy, along with some easy properties. Sections 3 and 4 deal with the continuity and PWA nature of the control law. The boundaries of the regions are treated in Section 5. Finally, the results presented in this brief are illustrated with an example in Section 6.

2. Min–max MPC with bounded additive uncertainties

Consider the following state-space model with bounded additive uncertainties (Camacho & Bordóns, 2004):

$$x(t+1) = Ax(t) + Bu(t) + D\theta(t), \quad y(t) = Cx(t) \tag{1}$$

with $x(t) \in \mathbb{R}^{dim\, x}$, $u(t) \in \mathbb{R}^{dim\, u}$, $\theta(t) \in \{\theta \in \mathbb{R}^{dim\, \theta} : \|\theta\|_\infty \leqslant \theta_m\}$, $y(t) \in \mathbb{R}^{dim\, y}$. Consider a sequence $\mathbf{u} = [u(t) \ldots u(t+N_u-1)]^{\mathrm{T}}$ of values of the control signal over a control horizon N_u and $\boldsymbol{\theta} = [\theta(t+1) \ldots \theta(t+N)]^{\mathrm{T}}$ a sequence of future values of $\theta(t)$ over a prediction horizon N. Furthermore, let $J(\boldsymbol{\theta}, \mathbf{u}, x)$ be a quadratic performance index of the form:

$$\begin{aligned} J(\boldsymbol{\theta}, \mathbf{u}, x) = {} & x(t+N|t, \boldsymbol{\theta})^{\mathrm{T}} P x(t+N|t, \boldsymbol{\theta}) \\ & + \sum_{j=1}^{N-1} x(t+j|t, \boldsymbol{\theta})^{\mathrm{T}} Q_j x(t+j|t, \boldsymbol{\theta}) \\ & + \sum_{j=0}^{N_u-1} u(t+j)^{\mathrm{T}} L_j u(t+j), \end{aligned} \tag{2}$$

where $x(t+j|t, \boldsymbol{\theta})$ is the prediction of the state for $t+j$ made at t when the future values of the uncertainty are supposed to be given by the sequence $\boldsymbol{\theta}$. When $N_u < N$ it is assumed that the control signal is constant and equal to $u(N_u-1)$ for $j = N_u, \ldots, N$. On the other hand $P, Q_j \in \mathbb{R}^{dim\, x \times dim\, x}$, $L_j \in \mathbb{R}^{dim\, u \times dim\, u}$ are symmetric positive definite matrices used as weighting parameters.

At any time, the state x and the sequence $\mathbf{u}$ must satisfy a set of nc affine constraints, such that only the pairs

$$(\mathbf{u}, x) : R_i^{\mathrm{T}} \mathbf{u} + \Gamma_i^{\mathrm{T}} \boldsymbol{\theta} \leqslant g_i + F_i^{\mathrm{T}} x, \quad i = 1, \ldots, nc \ \ \forall \boldsymbol{\theta} \in \boldsymbol{\Theta} \tag{3}$$

are admissible, where $\boldsymbol{\Theta} = \{\boldsymbol{\theta} \in \mathbb{R}^{N \cdot dim\, \theta} : \|\boldsymbol{\theta}\|_\infty \leqslant \theta_m\}$, $R_i \in \mathbb{R}^{(N_u \cdot dim\, u)}$, $F_i \in \mathbb{R}^{dim\, x}$, $\Gamma_i \in \mathbb{R}^{(N \cdot dim\, \theta)}$ and $g_i \in \mathbb{R}$. These constraints may arise from operational constraints or be used to guarantee stability. Note that

$$\max_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \Gamma_i^{\mathrm{T}} \boldsymbol{\theta} = \max_{\|\boldsymbol{\theta}\| \leqslant \theta_m} \Gamma_i^{\mathrm{T}} \boldsymbol{\theta} = \theta_m \|\Gamma_i\|_1,$$

where $\|\Gamma_i\|_1$ is the 1-norm of Γ_i, i.e., the sum of the absolute value of its components. Thus, the robust fulfillment of the constraints (3) is satisfied if and only if $R_i^{\mathrm{T}} \mathbf{u} + \theta_m \|\Gamma_i\|_1 \leqslant g_i + F_i^{\mathrm{T}} x, \quad i = 1, \ldots, nc$ (Alamo, Muñoz de la Peña, Limón Marruedo, & Camacho, 2005a). Therefore, robust constraint satisfaction of (3) is guaranteed by considering the following set of affine constraints:

$$R\mathbf{u} \leqslant c_\theta + Fx, \tag{4}$$

where matrices $R \in \mathbb{R}^{nc \times (N_u \cdot dim\, u)}$ and $F \in \mathbb{R}^{nc \times dim\, x}$ are composed of the row vectors R_i^{T} and F_i^{T} and the ith component of vector $c_\theta \in \mathbb{R}^{nc}$ is given by $g_i - \theta_m \|\Gamma_i\|_1$.

MMMPC (Campo & Morari, 1987) is based on finding the control correction sequence $\mathbf{u}$ that minimizes $J(\boldsymbol{\theta}, \mathbf{u}, x)$ for the worst possible case of the predicted future evolution of the process state or output signal. This is accomplished by the solution of a min–max problem such as

$$\begin{aligned} \mathbf{u}^*(x) = \operatorname*{argmin}_{\mathbf{u} \in \mathbf{U}} \quad & J^*(\mathbf{u}, x) \\ \text{s.t.} \quad & R\mathbf{u} \leqslant c_\theta + Fx, \end{aligned} \tag{5}$$

with

$$J^*(\mathbf{u}, x) = \max_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} J(\boldsymbol{\theta}, \mathbf{u}, x),$$

where $\mathbf{U} \subseteq \mathbb{R}^{N_u \cdot dim\, u}$ is compact. Of all possible values of x only those feasible are considered: that is, those belonging to

$$K^* \triangleq \{x \in \mathbb{R}^{dim\, x} : \exists \mathbf{u} \in \mathbf{U},\, R\mathbf{u} \leqslant c_\theta + Fx\}. \tag{6}$$

The solution of problem (5) is applied in a feedback manner using a receding horizon strategy (Camacho & Bordóns, 2004). Note that $J^*(\mathbf{u}, x)$ is the pointwise maximum of a set of an infinite number of quadratic cost functions of $\mathbf{u}$ and x. Thus, $J^*(\mathbf{u}, x)$ is a piecewise quadratic function of $\mathbf{u}$ and x. Note that a polytopic terminal constraint devised to provide robust stability can also be included within the constraints $R\mathbf{u} \leqslant c_\theta + Fx$ (see Mayne, Rawlings, Rao, & Scokaert, 2000 and references therein). Also a stabilizing terminal cost can also be considered via a proper choice for matrix P (Mayne et al., 2000).

Problem (5) is of the open-loop type. However, the results presented in this paper are also valid when using a semi-feedback approach (Mayne, 2001) in which the control input is given by $u(t) = -Kx(t) + v(t)$ where the feedback matrix K is chosen to achieve a certain desired property such as nominal stability or LQR optimality. The MMMPC controller will