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Brief paper

Diagonal stability of a class of cyclic systems and its connection with the secant criterion☆

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Abstract

We consider a class of systems with a cyclic interconnection structure that arises, among other examples, in dynamic models for certain biochemical reactions. We first show that a "secant" criterion for local stability, derived earlier in the literature, is in fact a necessary and sufficient condition for diagonal stability of the corresponding class of matrices. We then revisit a recent generalization of this criterion to output strictly passive systems, and recover the same stability condition using our diagonal stability result as a tool for constructing a Lyapunov function. Using this procedure for Lyapunov construction we exhibit classes of cyclic systems with sector nonlinearities and characterize their global stability properties.

Keywords: Diagonal stability; Vector Lyapunov functions; Passivity; Storage functions; Biochemical reactions

1. Introduction

In this paper we study systems characterized by a *cyclic* interconnection structure as depicted in Fig. 1. An important example where this structure arises is a sequence of biochemical reactions where the end product drives the first reaction as described by the model

$$\begin{aligned}
\dot{\chi}_1 &= -f_1(\chi_1) + g_n(\chi_n),\\
\dot{\chi}_2 &= -f_2(\chi_2) + g_1(\chi_1),\\
&\vdots\\
\dot{\chi}_n &= -f_n(\chi_n) + g_{n-1}(\chi_{n-1}).
\end{aligned} \tag{1}$$

Tyson and Othmer (1978) and Thron (1991) addressed the situation where $f_i(\cdot)$, $i=1,\ldots,n$, and $g_i(\cdot)$, $i=1,\ldots,n-1$ are increasing functions and $g_n(\cdot)$ is a decreasing function, which means that the intermediate products "facilitate" the next reaction while the end product "inhibits" the rate of the first reaction. To evaluate local stability properties of such reactions Tyson and Othmer (1978) and Thron (1991) analyzed the Jacobian linearization at the equilibrium, which is of the form

$$A = \begin{bmatrix} -\alpha_1 & 0 & \cdots & 0 & -\beta_n \\ \beta_1 & -\alpha_2 & \ddots & & 0 \\ 0 & \beta_2 & -\alpha_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \beta_{n-1} & -\alpha_n \end{bmatrix}, \quad \alpha_i > 0, \quad \beta_i > 0 \tag{2}$$

$i = 1,\ldots,n$, and showed that it is Hurwitz if

$$\frac{\beta_1 \ldots \beta_n}{\alpha_1 \ldots \alpha_n} < \sec(\pi/n)^n. \tag{3}$$

Unlike a *small-gain* condition which would restrict the right-hand side of (3) to be 1, criterion (3) also exploits the phase of the loop and allows the right-hand side to be 8 when $n = 3$, 4 when $n = 4$, 2.8854 when $n = 5$, etc. Furthermore, when α_i's are equal, (3) is also necessary for stability.

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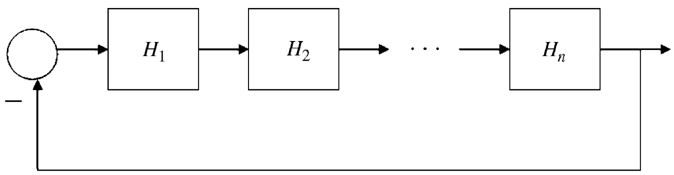

Fig. 1. A cyclic feedback interconnection of systems $H_1, \ldots, H_n$.

The objective of this paper is to extend this stability criterion to classes of nonlinear systems, including (1), by building on a *passivity* interpretation presented recently in Sontag (2006). We first revisit Sontag (2006), which derived an analog of (3) when the blocks in Fig. 1 are *output strictly passive* (Sepulchre, Janković, & Kokotović, 1997; van der Schaft, 2000), and recover the same stability result with a Lyapunov proof that complements the input–output arguments in Sontag (2006). Our Lyapunov function consists in a weighted sum of *storage functions* for each block, with the weights selected judiciously according to a diagonal stability result proved in this paper for the class of matrices (2). This construction resembles the method of *vector Lyapunov functions* in the literature of large-scale systems (Michel & Miller, 1977; Šiljak, 1978), where a Lyapunov function is assembled from a weighted sum of several components.

We next study the case where some of the blocks in Fig. 1 are static *sector* nonlinearities. When such a nonlinearity is time invariant and preceded by a linear, first-order, dynamic block we relax our stability criterion with a special Lyapunov construction that mimics the proof of the Popov criterion (Khalil, 2002). We next apply a similar construction to system (1), and extend the secant condition (3) to become a criterion for global asymptotic stability. Our main assumption in this result is that $f_i(\cdot)$'s and $g_i(\cdot)$'s satisfy a *sector* property, and that the growth ratio of $g_i(\cdot)$ relative to $f_i(\cdot)$ be bounded by a constant that plays the role of β_i/α_i in (3). The next result extends this condition to the case where the state variables are nonnegative quantities as in biochemical reactions.

The results of this paper previewed above all hinge upon our key theorem for diagonal stability of (2), presented in Section 2. Using this theorem, Section 3 studies the cyclic interconnection in Fig. 1, and gives a procedure for selecting the weights in our Lyapunov function construction from storage functions. Section 4 derives a Popov-type relaxed stability criterion for static, time-invariant, sector nonlinearities. Section 5 revisits system (1) and proves global asymptotic stability. Section 6 extends this result to systems with nonnegative state variables. An independent result in Section 7 studies a cascade of output strictly passive systems, and uses our main theorem on diagonal stability to prove an *input feedforward passivity* (IFP) (Sepulchre et al., 1997) property for the cascade, which quantifies the amount of feedforward gain required to re-establish passivity.

2. Main theorem for diagonal stability

The key ingredient for all of the results in this paper is Theorem 1, which states that (3) is a necessary and sufficient condition for *diagonal stability* of (2). This theorem is of independent interest because existing results for diagonal stability of various classes of matrices, such as those surveyed in Redheffer (1985) and Kaszkurewicz and Bhaya (2000) do not address the cyclic structure exhibited by (2). In particular, the sign reversal for β_n in (2) rules out the "M-matrix" condition, which is applicable when all off-diagonal terms are nonnegative.

Theorem 1. *The matrix* (2) *is diagonally stable*; *that is*, *it satisfies*

$$DA + A^{\mathrm{T}}D < 0 \tag{4}$$

for some diagonal matrix $D > 0$, *if and only if* (3) *holds.*

The remaining results of this paper are presented in the form of corollaries to this theorem. Tyson and Othmer (1978) and Thron (1991) studied the characteristic polynomial of (2) and showed that (3) is a sufficient condition for A to be Hurwitz. They further showed that this condition is also necessary when α_i's are equal. Theorem 1 proves that (3) is necessary and sufficient for *diagonal* stability even when α_i's are not equal. This means that if A is Hurwitz but (3) fails, then the Lyapunov inequality $A^{\mathrm{T}}P + PA < 0$ does not admit a diagonal solution.

Proof of Theorem 1. We prove the theorem for the matrix

$$A_0 = \begin{bmatrix} -1 & 0 & \cdots & 0 & -\gamma_1 \\ \gamma_2 & -1 & \ddots & & 0 \\ 0 & \gamma_3 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \gamma_n & -1 \end{bmatrix} \tag{5}$$

because other matrices of the form (2) can be obtained by scaling the rows of this A_0 by positive constants, which does not change diagonal stability. Our task is thus to prove necessity and sufficiency for diagonal stability of the condition

$$\gamma_1 \ldots \gamma_n < \sec(\pi/n)^n, \tag{6}$$

which is (3) for A_0. Necessity follows because the diagonal entries of A_0 are equal, in which case (6) is necessary for A_0 to be Hurwitz (Tyson & Othmer, 1978). To prove that (6) is also sufficient for diagonal stability, we define

$$r := (\gamma_1 \ldots \gamma_n)^{1/n} > 0,$$
$$\Delta := \mathrm{diag}\left\{1, -\frac{\gamma_2}{r}, \frac{\gamma_2\gamma_3}{r^2}, \ldots, (-1)^{n+1}\frac{\gamma_2 \cdots \gamma_n}{r^{n-1}}\right\} \tag{7}$$

and note that

$$-\Delta^{-1}A_0\Delta = \begin{bmatrix} 1 & 0 & \cdots & 0 & (-1)^{n+1}r \\ r & 1 & \ddots & & 0 \\ 0 & r & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & r & 1 \end{bmatrix}. \tag{8}$$

Thus, with the choice

$$D = \Delta^{-2} \tag{9}$$