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Constructive solutions to spectral and inner–outer factorizations with respect to the disk☆

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Abstract

We use state-space realizations to solve the spectral and inner–outer factorization problems with respect to the unit disk formulated for a completely general rational matrix. The algorithm is based on orthogonal transformations and standard reliable procedures for solving Stein and Riccati equations. The formulas apply in particular to general descriptor systems or polynomial matrices. The main novelty is that we allow for arbitrary rank, poles and zeros on the unit circle, or at infinity.

Keywords: Discrete-time systems; Inner–outer factorization; Spectral factorization; Riccati equations; Rational matrices

1. Introduction

Throughout the paper we consider rational matrices with real coefficients as this is the typical case in control applications. For a $p \times m$ rational matrix $\Phi(z)$ with real coefficients and of the complex variable z we define the para-Hermitian conjugate (with respect to the unit circle) to be the $m \times p$ rational matrix $\Phi^*(z)=\Phi^{\mathrm{T}}(1/z)$, where the upper index T stands for the transpose. A square, rational matrix Φ is said to be para-Hermitian if $\Phi^*(z)=\Phi(z)$. A $p \times m$ rational matrix Φ is called marginally stable if it is analytic outside the closed unit disk and at infinity, and is called stable if it is marginally stable and analytic on the unit circle. We denote the open unit disk and its closure by $\mathbb{D}$ and $\overline{\mathbb{D}}$, respectively.

It is well-known that an $m \times m$ para-Hermitian matrix Φ with (normal) rank r has a factorization of the form $\Phi(z)=\phi^*(z)\phi(z)$ for some $r \times m$ rational matrix ϕ if and only if Φ is positive semi-definite at almost every point on the unit circle. Such a factorization is in general non-unique and one may add additional requirements for the factor ϕ. This basic result is known as *spectral factorization* in the engineering literature. A standard reference for spectral factorization is Youla (1961).

In this paper, we deal with a specialized case of this general spectral factorization problem in which Φ is pre-factorized as $\Phi(z)=G^*(z)G(z)$, where G is a given arbitrary real rational matrix, and the problem is to compute various relevant spectral factors (not always of dimension $r \times m$). More precisely, we consider the following *spectral factorization problem*: Given an arbitrary real rational $p \times m$ matrix $G(z)$ of normal rank r, find a rational matrix $\phi(z)$, such that

$$G^*(z)G(z)=\phi^*(z)\phi(z), \tag{1}$$

where the structural elements of $\phi(z)$ (poles, zeros, minimal indices, normal rank) must satisfy some additional requirements. $\phi(z)$ is called a spectral factor and (1) defines a spectral factorization of $G^*(z)G(z)$.

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This spectral factorization problem with ϕ of dimension $r \times m$ is related to another type of factorization

$$G(z) = \theta(z)\phi(z), \tag{2}$$

where θ is required $p \times r$ isometric (i.e., satisfies $\theta^*(z)\theta(z) = I$) and ϕ is the same spectral factor as in (1) required to have a particular location of poles and/or zeros. In particular, if G is marginally stable, θ is required isometric and stable and ϕ is required marginally stable then the factorization (2) is known as *inner–outer*, θ being the inner and ϕ the outer factors. The two factorization problems (1) and (2) are strongly connected in the following sense. Let (1) be a spectral factorization of G, with θ of dimension $r \times m$. Then it is easy to check that $\theta := G\phi^{\#}$ is isometric, where $\phi^{\#}$ is a right inverse of ϕ, and θ together with ϕ define a factorization (2) of G. Conversely, if (2) is a factorization of G with θ isometric, then we immediately find that ϕ is a spectral factor of G satisfying (1).

There are various constraints that one may impose such as to customize the spectral factors for various applications and reduce the class of possible solutions. In electrical engineering most interesting factorizations of type (1) require ϕ to be (*marginally*) *stable* (i.e. analytic outside the closed unit disk including at infinity) or *minimum phase* (i.e. of full row rank outside the closed unit disk including at infinity), or both, but all can accommodate within the framework introduced in this paper. For example, assume G is an arbitrary polynomial matrix. Then the spectral factorization problem requires a polynomial matrix ϕ of full row rank and having all zeros in $\overline{\mathbb{D}}$ such that (1) holds. For a detailed treatment of various factorization problems for linear systems and operators the reader is referred to Fuhrmann (1981).

Spectral factorizations of type (1) appear throughout in control systems, identification, signal processing, network and circuit theory (see for example Lindquist, Michaletzky, & Picci, 1995; Ferrante, Pavon, & Pinzoni, 2003). Since many fundamental problems in these branches can be solved once the spectral factors are known, a wealth of research efforts has been invested in their construction and finding their various properties. However, all approaches proposed so far fail short to effectively construct the spectral factor if G has not full column rank and has poles/zeros on the unit circle or at infinity (see Gu, Tsai, O'Young, & Postlethwaite, 1989; Zhang & Freudenberg, 1992; Katayama, 1996, and for a historical perspective, Oară & Varga, 2000). One of the most general methods available for discrete-time systems can deal with G of arbitrary rank but without zeros on the unit circle (Ionescu & Oară, 1996; Ionescu, Oară, & Weiss, 1999). However, if such zeros are present, semi-stabilizing solutions instead of stabilizing solutions to Riccati equations have to be computed and this implies a symmetric separation of the unit circle eigenvalues of the associated symplectic pencil. Unfortunately, no numerical method to cope efficiently with this task is available. Other methods which deal with zeros on the unit disk but apply for a full row rank G without zeros at infinity were proposed in Van Dooren (1990) and Varga (1998). The most general solution available so far is Dewilde and van der Veen (2000) where the discrete-time inner–outer factorization problem is considered in a quite different context of inverting locally finite systems of equations. In Dewilde and van der Veen (2000) they solve the minimum phase problem with a couple of restrictions that can be handled only by bilinear transformations: absence of poles and zeros at infinity and absence of zeros at zero. However, the overall solution seems to us as unnecessarily intricate; more on this in Conclusions.

The corresponding inner–outer and spectral factorizations for continuous-time systems have been recently solved in Oară and Varga (2000) by using a method of successive poles/zeros/minimal indices dislocation with all-pass factors. In this paper, we extend these ideas such as to become applicable to the peculiarities of the discrete case. In contrast to the continuous-time case, we have to develop a different technical machinery that is able to cope with poles and zeros at infinity. A preliminary attempt to apply the same ideas to the discrete-time case is Oară and Varga (1999a). However, in Oară and Varga (1999a) the approach is via some implicit bilinear transformations which lead unnecessarily to descriptor realizations of the spectral factor and solutions to nonstandard Riccati equations.

We report here the genuine discrete-time solution of the spectral factorization problem with respect to the unit disk by deriving formulas for the spectral factor in state-space form. Moreover, we eliminate the implicit recursiveness in Oară and Varga (2000) and show that once the appropriate projections are taken, the spectral factor can be written directly in terms of the solutions to a standard Stein and a standard Riccati equation. This is done in much similar a way to what is already well known in the literature in pretty particular cases. Finally, we work under the most relaxed possible assumptions on the realization of G.

Specifically, we give state-space formulas and numerically reliable procedures to compute the solutions to the spectral factorization problems (1) and (2) in three cases corresponding to one of the following additional requirements on the spectral factor:

(MP) ϕ has full row rank outside the closed unit disk including at infinity (is *minimum phase*);

(S) ϕ is analytic outside the closed unit disk including at infinity (is *marginally stable*);

(SMP) ϕ is analytic and has full row rank outside the closed unit disk including at infinity (is marginally stable and minimum phase).

We use terms from the control jargon to name these three factorization problems by the corresponding feature of the spectral factor: minimum phase (MP), stable (S), and stable and minimum phase (SMP).

In order to derive formulas for the spectral factors, we implicitly make use of the equivalence between (1) and (2). For example, in the case of the MP factorization ϕ is an $r \times m$ rational matrix having no zeros outside the closed unit disk, while θ is $p \times r$, stable and isometric. Hence