PERGAMON

Automatica 38 (2002) 2015–2019

automatica

www.elsevier.com/locate/automatica

Technical Communique

Absolute stability criteria with prescribed decay rate for finite-dimensional and delay systems☆

Pierre-Alexandre Bliman

I.N.R.I.A., Rocquencourt B.P. 105, 78153 Le Chesnay Cedex, France

Received 9 August 2001; received in revised form 11 February 2002; accepted 13 May 2002

Abstract

We provide here an extension of Popov criterion, permitting to check exponential stability with prescribed decay rate (otherwise called α-stability) of nonlinear delay systems with sector-bounded nonlinearities. As for the celebrated result, the main hypothesis is expressed under a frequency form. For the delay-free case, the latter is equivalent to a linear matrix inequality, whose solution may be found by widespread algorithms. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: α-stability; Absolute stability; Popov criterion; Delay analysis; Frequency domains; Linear matrix inequality; Lyapunov function

1. Introduction

Asymptotic stability of the controlled systems is usually not sufficient, one also requires a minimal prescribed decay rate. Results in this direction have been proposed for linear delay systems, see Bourlès (1987), Niculescu (1998) and Wang and Wang (1996) and the references therein. Also, a result has been obtained for nonlinear delay systems (Sun & Hsieh, 1998). In the present paper, we provide such a result for nonlinear finite-dimensional or delay systems with sector restricted nonlinearities.

We consider more precisely multivariable nonlinear control systems given by one of the following differential and functional differential equations:

$$\dot{x} = Ax + Bu, \quad x(0) = \phi \in \mathbb{R}^n,$$
$$u = -\psi(y), \quad y = Cx, \tag{1a}$$

$$\dot{x} = \sum_{l=0}^{L} A_l x(t - h_l) + Bu, \quad x|_{[-h,0]} = \phi \in \mathscr{C}([-h, 0]),$$
$$u = -\psi(y), \quad y = \sum_{l=0}^{L} C_l x(t - h_l), \tag{1b}$$

where $n, p, L \in \mathbb{N}$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $A, A_l \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C, C_l \in \mathbb{R}^{p\times n}$, $0 \leqslant h_0 < \ldots < h_L$, $h \stackrel{\text{def}}{=} \max\{h_l\}$. One denotes by H the matrix transfer function corresponding to the system under study, namely:

$$H(s) = C(sI - A)^{-1}B, \tag{2a}$$

$$H(s) = \left(\sum_{l=0}^{L} C_l e^{-h_l s}\right)\left(sI - \sum_{l=0}^{L} A_l e^{-h_l s}\right)^{-1} B. \tag{2b}$$

Let the nonlinearity $\psi : \mathbb{R}^+ \times \mathbb{R}^p \to \mathbb{R}^p$ be time-independent, *decentralized* (Khalil, 1992) ($\forall i \in \{1, \ldots, p\}$, $\psi_i(y) = \psi_i(y_i)$), and fulfil the following *sector condition*:

$$\forall y \in \mathbb{R}^p, \quad \psi(y)^{\mathrm{T}}(\psi(y) - Ky) \leqslant 0, \tag{3}$$

for a certain diagonal matrix $K \geqslant 0$ (the inequality is hence also valid componentwise). By analogy with the concepts of absolute and of α-stability (Niculescu, 1998; Sun & Hsieh, 1998) we define:

Definition 1. Let α be a nonnegative scalar. System (1a) (resp. (1b)) is called *absolutely stable with decay rate* α (or *absolutely* α*-stable*) if, for any global solution x,

$$\lim_{t\to+\infty} \frac{e^{\alpha t} x(t)}{\|\phi\|} = 0,$$

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Rodolphe Sepulchre under the direction of Editor Paul Van den Hof.

E-mail address: pierre-alexandre.bliman@inria.fr (P.-A. Bliman).

PII: S0005-1098(02)00092-4

where the convergence is uniform w.r.t the initial condition $\phi \neq 0$ and to the nonlinearity ψ fulfilling (3). Here, $\|\cdot\|$ denotes the euclidian norm in $\mathbb{R}^n$ (resp. the uniform convergence norm in $\mathscr{C}([-h,0];\mathbb{R}^n)$).

In the present note, some simple absolute α-stability criteria are proposed. The presentation is unified for both finite-dimensional and delay systems, of type (1a) and (1b), respectively. Our main contribution is the following:

- A criterion for finite-dimensional systems (1a) is provided (Theorem 2). It is expressed equivalently as a linear matrix inequality (LMI), a standard class of problems for which sound numerical methods have been developed (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994), or under frequency domain form.
- The previous frequency domain form of the criterion is shown to be valid for delay systems (1b) too (Theorem 5). The results generalize Popov criterion, which is found when taking $\alpha = 0$. Notice that one may easily adapt the results proposed herein to absolute α-stability for systems with *time-varying* nonlinearities, or to local stability results, see Bliman (2000). More generally, the results stated here could be applied to more general systems (e.g. systems with distributed delays, integral systems), as it is indeed the case for Popov criterion, see Halanay (1966), Section 4.6. and Corduneanu (1972).

The case of finite-dimensional systems is treated in Section 2. The frequency domain criterion for delay systems stability is stated in Section 3. Section 4 is a conclusion. The issues of existence and uniqueness of the solutions are not considered, as they have been extensively studied. In all the sequel, it is assumed that there exist *global solutions of* (1a) (resp. (1b)), that is, *by definition*: for all $\phi \in \mathbb{R}^n$ (resp. for all $\phi \in \mathscr{C}([-h,0];\mathbb{R}^n)$), there exists a continuous function x defined on $[0,+\infty)$ (resp. on $[-h,+\infty)$), absolutely continuous (AC) on $[0,+\infty)$, such that $x(0)=\phi$ (resp. $x|_{[-h,0]} = \phi$) and (1a) (resp. (1b)) is fulfilled almost everywhere on $[0,+\infty)$.

Notations. The abbreviation SPR is used for "strictly positive real". For $z \in \mathbb{R}$, one denotes by $\operatorname{sgn} z$ the sign of z ($\operatorname{sgn} 0 = 1$ or -1 indifferently), and $|z|_+ \stackrel{\text{def}}{=} \sup\{z,0\}$, $|z|_- \stackrel{\text{def}}{=} \sup\{-z,0\}$. By convention, one extends the action of any map acting on scalar or scalar-valued functions to an operator acting on matrices or matrix-valued functions, obtained by componentwise application. As an example, for any diagonal matrix η, $|\eta|_\pm = \sup\{\pm\eta, 0\} = \operatorname{diag}\{\sup\{\pm\eta_i,0\}\} = \operatorname{diag}\{|\eta_i|_\pm\}$.

2. Finite-dimensional systems

Theorem 2. *Assume that*

(H) *The nonlinearity ψ is measurable, decentralized and there exists a diagonal matrix $K \stackrel{\text{def}}{=} \operatorname{diag}\{K_i\} \geqslant 0$ such that* (3) *holds.*

$$R \stackrel{\text{def}}{=} R(P,\eta) = \begin{pmatrix} A^{\mathrm{T}}P + PA + 2\alpha P + 2\alpha C^{\mathrm{T}}K|\eta|_+KC & -PB + C^{\mathrm{T}}K + A^{\mathrm{T}}C^{\mathrm{T}}K\eta \\ -B^{\mathrm{T}}P + KC + \eta KCA & -2I - \eta KCB - B^{\mathrm{T}}C^{\mathrm{T}}K\eta \end{pmatrix}. \quad (4)$$

Let $\alpha \geqslant 0$, and η be a diagonal matrix in $\mathbb{R}^{p\times p}$. The following properties are equivalent, and imply absolute α-stability of system (1a).

- *There exists a symmetric definite positive matrix $P \in \mathbb{R}^{n\times n}$ such that $R(P,\eta)$ defined in* (4) *is definite negative.*
- *The roots of* $\det(sI - A)$ *have real part smaller than $-\alpha$, and for H defined in* (2a), *the transfer*

$$I + (I + \eta(s-\alpha)I))KH(s-\alpha) - \alpha H^{\mathrm{H}}(s-\alpha)K|\eta|_+KH(s-\alpha) \text{ is SPR.} \quad (5)$$

Theorem 2 extends, for finite-dimensional systems, the usual form of the Popov criterion, obtained for $\alpha = 0$. The result of α-stability obtained for null Popov slope η constitutes an extension of the circle criterion, and may be found in Naumov and Tsypkin (1964). Given the sign of the components of the diagonal matrix η, (4) is a linear matrix inequality in the two unknowns P, η.

Proof of Theorem 2. The following lemma plays central role in the demonstration of Theorems 2 and 5.

Lemma 3. *Let $i \in \{1,\dots,p\}$, $\mathscr{Y}_i : [0,+\infty) \to \mathbb{R}$ be differentiable a.e. Then, for almost any $T \geqslant 0$,*

$$\frac{\mathrm{d}}{\mathrm{d}T}\left[\mathrm{e}^{2\alpha T} \int_0^{\mathrm{e}^{-\alpha T}\mathscr{Y}_i(T)} \psi_i(z)\,\mathrm{d}z \right] \leqslant \alpha K_i \mathscr{Y}_i^2(T) + \mathrm{e}^{\alpha T}(\dot{\mathscr{Y}}_i(T) - \alpha\mathscr{Y}_i(T))\psi_i(\mathrm{e}^{-\alpha T}\mathscr{Y}_i(T)).$$

Proof. The left-hand side being equal to $2\alpha \mathrm{e}^{2\alpha T}\int_0^{\mathrm{e}^{-\alpha T}\mathscr{Y}_i(T)} \psi_i(z)\,\mathrm{d}z + \mathrm{e}^{\alpha T}(\dot{\mathscr{Y}}_i(T) - \alpha\mathscr{Y}_i(T))\psi_i(\mathrm{e}^{-\alpha T}\mathscr{Y}_i(T))$, bound the integral term, using the fact that $\forall z \in \mathbb{R}$, $\int_0^z \psi_i(z')\,\mathrm{d}z' \leqslant \frac{1}{2}K_i z^2$, due to sector condition. □

- Assume first that $\eta \geqslant 0$. With the change of variables $\mathscr{X}(t) \stackrel{\text{def}}{=} \mathrm{e}^{\alpha t}x(t)$, $\mathscr{Y}(t) \stackrel{\text{def}}{=} \mathrm{e}^{\alpha t}y(t)$, $\varphi(t,\mathscr{Y}) \stackrel{\text{def}}{=} \mathrm{e}^{\alpha t}\psi(\mathrm{e}^{-\alpha t}\mathscr{Y})$, (1a) is changed into the *nonautonomous* system

$$\dot{\mathscr{X}} = (\alpha I + A)\mathscr{X}(t) - B\varphi(t,\mathscr{Y}(t)), \quad \mathscr{Y} = C\mathscr{X}. \quad (6)$$