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Nonlinear water waves in shallow water in the presence of constant vorticity: A Whitham approach

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ABSTRACT

Two-dimensional nonlinear gravity waves travelling in shallow water on a vertically sheared current of constant vorticity are considered. Using Euler equations, in the shallow water approximation, hyperbolic equations for the surface elevation and the horizontal velocity are derived. Using Riemann invariants of these equations, that are obtained analytically, a closed-form nonlinear evolution equation for the surface elevation is derived. A dispersive term is added to this equation using the exact linear dispersion relation. With this new single first-order partial differential equation, vorticity effects on undular bores are studied. Within the framework of weakly nonlinear waves, a KdV-type equation and a Whitham equation with constant vorticity are derived from this new model and the effect of vorticity on solitary waves and periodic waves is considered. Furthermore, within the framework of the new model and the Whitham equation a study of the effect of vorticity on the breaking time of dispersive waves and hyperbolic waves as well is carried out.

1. Introduction

Generally, in coastal and ocean waters, current velocity profiles are established by bottom friction and wind stress at the sea surface, and consequently are vertically varying. Ebb and flood currents due to the tide may have an important effect on water wave properties. In any region where the wind blows, the generated current affects the behaviour of the waves. The present work focuses on the nonlinear evolution of two-dimensional gravity waves propagating in shallow water on a shear current which varies linearly with depth. Consequently, the waves are travelling on a flow of constant vorticity. Considering constant vorticity is an approximation which allows the analytical derivation of a unique partial differential equation governing the nonlinear evolution of hyperbolic or dispersive waves in shallow water.

Within the framework of long waves, there are very few papers devoted to unsteady nonlinear evolution of water waves propagating on an underlying vertically sheared current. Freeman & Johnson [1] derived a KdV equation governing the time evolution of gravity waves on shear flows of arbitrary vorticity distribution. Using an asymptotic expansion method Choi [2] derived a Green–Naghdi system for long gravity waves in uniform shear flows (constant vorticity) and for weakly nonlinear waves he deduced from this system a Boussinesq-type equation and a KdV equation. The derivation of a Camassa–Holm equation for water waves travelling over a linear shear flow was carried out by Johnson [3] who used a multiple-scale method. Ivanov [4] and later Escher et al. [5] derived several two-component integrable systems in the presence of constant vorticity, including an extended version of the Camassa–Holm equation, which approximate the Euler equation for an incompressible fluid. They emphasised that these models may capture the onset of wave breaking in finite-time. Johnson [6] derived a Boussinesq type equation with constant vorticity. Very recently, Richard & Gavrilyuk [7] derived a dispersive shallow water model which is a generalisation of the classical Green–Naghdi model to the case of shear flows. Castro & Lannes [8] investigated rigorously a Green–Naghdi type system including a general vorticity. At the same time and separately Kharif et al. [9] and Hur [10] have derived shallow water wave equations with constant vorticity. Kharif et al. [9] have considered the effect of a vertically sheared current on rogue wave properties whereas Hur [10] investigated the effect of vorticity on modulational instability and onset of breaking. Independently and concurrently, the shallow-water equations with constant vorticity were derived by Bjornestad & Kalisch [11], too. They investigated long water waves travelling over a shear flow towards a sloping beach.

In the coastal zone where the vorticity is an important ingredient, water wave dynamics is governed by the Euler equations with boundary conditions at the free surface which is nonlinear and unknown *a priori*. Numerical integration of this system of equations is not a trivial task and so more simple models have been derived in the past to describe and investigate the dynamics of water waves phenomena in shallow water such as undular bores

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and nonlinear long wave propagation. For a review one can refer to the paper by Lannes & Bonneton [12].

Following Whitham [13], we propose a new model derived from the Euler equation for water waves propagating on a vertically sheared current of constant vorticity in shallow water. This new approach is easier to handle numerically. The heuristic introduction of dispersion allows the study of strongly nonlinear two-dimensional long gravity waves in the presence of vorticity and as well that of undular bores. From this model we derive, within the framework of weakly nonlinear waves satisfying the exact linear dispersion a Whitham equation and for weakly nonlinear and weakly dispersive waves the KdV equation previously obtained by Freeman & Johnson [1] and Choi [2]. These different equations are then used to investigate the effect of constant vorticity on breaking time of dispersive waves and hyperbolic waves as well.

2. Derivation of the new approach: The generalised Whitham equation

We consider two-dimensional gravity water waves propagating at the free surface of a vertically sheared current of uniform intensity Ω which is the opposite of the vorticity. The wave train moves along the $x-$ axis and the $z-$ axis is oriented upward opposite to the gravity. The origin $z = 0$ is the undisturbed free surface and $z = -h(x)$ is the rigid bottom.

The continuity equation is

$$u_x + w_z = 0 \tag{1}$$

where u and w are the longitudinal and vertical components of the wave induced velocity, respectively. The underlying current is $U = U_0 + \Omega z$ where U_0 is the constant surface velocity.

Integration of Eq. (1) gives

$$w(z=\eta) - w(z=-h) = -\int_{-h(x)}^{\eta(x,t)} u_x dz \tag{2}$$

where η is the surface elevation.

Note that

$$\int_{-h(x)}^{\eta(x,t)} u_x dz = \frac{\partial}{\partial x}\int_{-h(x)}^{\eta(x,t)} u dz - u(z=\eta)\eta_x - u(z=-h)h_x \tag{3}$$

The kinematic boundary condition at the free surface is

$$\eta_t + (u + U_0 + \Omega\eta)\eta_x - w = 0 \quad \text{on} \quad z = \eta(x,t) \tag{4}$$

The bottom boundary condition writes

$$(u + U_0 - \Omega h)h_x + w = 0 \quad \text{on} \quad z = -h(x) \tag{5}$$

We assume h constant, then

$$w = 0 \quad \text{on} \quad z = -h \tag{6}$$

From Eq. (4) it follows that

$$w(z=\eta) = \eta_t + (u + U_0 + \Omega\eta)_{z=\eta}\eta_x \tag{7}$$

$$w(z=\eta) = \eta_t + [u(z=\eta) + U_0 + \Omega\eta]\eta_x \tag{8}$$

Eq. (2) becomes

$$-\int_{-h}^{\eta(x,t)} u_x dz = \eta_t + [u(z=\eta) + U_0 + \Omega\eta]\eta_x \tag{9}$$

Using Eq. (3) with $h_x = 0$ we obtain

$$\frac{\partial}{\partial x}\int_{-h}^{\eta(x,t)} u dz + \eta_t + (U_0 + \Omega\eta)\eta_x = 0 \tag{10}$$

We assume u does not depend on z, then

$$\eta_t + \frac{\partial}{\partial x}[u(\eta + h) + \frac{\Omega}{2}\eta^2 + U_0\eta] = 0 \tag{11}$$

Eq. (11) corresponds to mass conservation in shallow water in the presence of constant vorticity.

Under the assumption of hydrostatic pressure, the Euler equation in x-direction is

$$u_t + (u + U_0 + \Omega z)u_x + \Omega w + g\eta_x = 0 \tag{12}$$

where g is the gravity.

Using the continuity equation and boundary conditions that w satisfies on the bottom and at the free surface, we obtain

$$w = -(z+h)u_x \tag{13}$$

It follows that the Euler equation becomes

$$u_t + (u + U_0 - \Omega h)u_x + g\eta_x = 0 \tag{14}$$

The dynamics of non dispersive shallow water waves on a vertically sheared current of constant vorticity is governed by Eqs. (11) and (14).

The pair of Eqs. (11) and (14) admits the following Riemann invariants

$$u + \frac{\Omega H}{2} \pm \left\{\sqrt{gH + \Omega^2H^2/4} + \frac{g}{\Omega}\ln\left[1 + \frac{\Omega}{2g}(\Omega H + 2\sqrt{gH + \Omega^2H^2/4})\right]\right\} = \text{constant}$$

on characteristic lines

$$\frac{dx}{dt} = u + U_0 + \frac{1}{2}\Omega(\eta - h) \pm \sqrt{gH + \frac{\Omega^2H^2}{4}} \tag{15}$$

where $H = \eta + h$.

The constant is determined for $u = 0$ and $\eta = 0$ or $H = h$.

Finally

$$u + \frac{\Omega\eta}{2} + \sqrt{gH + \Omega^2H^2/4} - \sqrt{gh + \Omega^2h^2/4} + \frac{g}{\Omega}\ln\left[\frac{1 + \frac{\Omega}{2g}(\Omega H + 2\sqrt{gH + \Omega^2H^2/4})}{1 + \frac{\Omega}{2g}(\Omega h + 2\sqrt{gh + \Omega^2h^2/4})}\right] = 0 \tag{16}$$

$$u + \frac{\Omega\eta}{2} - \sqrt{gH + \Omega^2H^2/4} + \sqrt{gh + \Omega^2h^2/4} - \frac{g}{\Omega}\ln\left[\frac{1 + \frac{\Omega}{2g}(\Omega H + 2\sqrt{gH + \Omega^2H^2/4})}{1 + \frac{\Omega}{2g}(\Omega h + 2\sqrt{gh + \Omega^2h^2/4})}\right] = 0 \tag{17}$$

Let us consider a wave moving rightwards

$$u = -\frac{\Omega\eta}{2} + \sqrt{gH + \Omega^2H^2/4} - \sqrt{gh + \Omega^2h^2/4} + \frac{g}{\Omega}\ln\left[\frac{1 + \frac{\Omega}{2g}(\Omega H + 2\sqrt{gH + \Omega^2H^2/4})}{1 + \frac{\Omega}{2g}(\Omega h + 2\sqrt{gh + \Omega^2h^2/4})}\right] \tag{18}$$

Substituting this expression into Eq. (11) gives Eq. (19) given in Box I. Eq. (19) is fully nonlinear and describes the spatio-temporal evolution of hyperbolic water waves in shallow water in the presence of constant vorticity. This equation is equivalent to the system of Eqs. (11) and (14) for waves moving rightwards.

Following Whitham [13], full linear dispersion is introduced heuristically (20) as in Box II. where $K * \eta_x$ is a convolution product. The kernel K is given as the inverse Fourier transform of the fully linear dispersion relation of gravity waves in finite depth in the presence of constant vorticity Ω: $K = F^{-1}(c)$ with

$$c = U_0 - \frac{\Omega\tanh(kh)}{2k} + \sqrt{\frac{g\tanh(kh)}{k} + \frac{\Omega^2\tanh^2(kh)}{4k^2}}$$