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On the initial evolution of the weak turbulence spectrum in a system with a decay dispersion relation

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ABSTRACT

The rate of change $\varepsilon(k,t) = \partial E(k,t)/\partial t$ of the energy spectrum $E(k,t)$ of a weak turbulence in a system with a decay dispersion relation is investigated with particular interests in the initial stage of evolution such as the first several tens of periods. The theoretical predictions given by the kinetic equation of Hasselmann and that of Janssen which have been derived by the weak turbulence theory are compared with the results of direct numerical simulation (DNS) with a great accuracy. It is shown that $\varepsilon(k,t)$ predicted by Janssen's equation correctly reproduces the rapid variation of $\varepsilon(k,t)$ with the linear time-scale which is observed in DNS, while that predicted by Hasselmann's equation does not. It is also shown that the transient behavior of the asymptotic approach of Janssen's $\varepsilon(k,t)$ to Hasselmann's one strongly depends on the wavenumber. For wavenumbers smaller than the spectral peak, the approach is exponential, while for wavenumbers larger than the spectral peak it contains a damped oscillation whose amplitude decays in time like $1/t$. A reasonable explanation for the origin of this damped oscillation is given in terms of the frequency mismatch of the three-wave "sum" interactions.

1. Introduction

In a weak turbulence, like the ocean wave field, many wave trains with different frequencies and directions of propagation interact with each other and exchange energy through the nonlinearity in the governing equations and/or the boundary conditions. For such a complex system a deterministic description in terms of the primitive variables such as the displacement of the free surface and the velocity potential is not practical, and a statistical description in terms of the energy spectrum or higher moments are reasonable [1–4]. For a weak turbulence of surface gravity waves, Hasselmann [5] has derived a kinetic equation which governs the evolution of the action spectrum $n(\boldsymbol{k},t)$ of the wave field, which is related to the energy spectrum $E(\boldsymbol{k},t)$ by $n(\boldsymbol{k},t) = E(\boldsymbol{k},t)/\omega(\boldsymbol{k})$, by using the so-called random-phase approximation. Here, $\omega(\boldsymbol{k})$ is the frequency of the wavenumber $\boldsymbol{k}$ given by the linear dispersion relation. According to Hasselmann's theory, wave action is exchanged only among those four waves which satisfy four-wave resonance conditions.

Afterward Janssen [6] has derived a generalized kinetic equation which also describes the evolution of $n(\boldsymbol{k},t)$ by a multiple time-scale analysis. In Janssen's equation the contribution from non-resonant interactions are also retained. Other types of generalized kinetic equations have also been derived in [7,8], and several studies have been done in view of the application of these generalized kinetic equations to practical ocean wave conditions [9–11]. When one takes the limit $t \to \infty$, Janssen's equation as well as other generalized kinetic equations reduces to Hasselmann's equation. However, the transient behavior of the asymptotic approach of Janssen's and other generalized kinetic equations to Hasselmann's equation has never been investigated quantitatively.

In the authors' previous work [12], we numerically investigated the behavior of a cubic Hamiltonian system with a decay dispersion relation in order to examine the validity of various theoretical predictions made by the random-phase-and-amplitude-formalism (RPA) which has been developed in [13]. In that work we found that the rate of change $\varepsilon(k,t) = \partial E(k,t)/\partial t$ of the energy spectrum $E(k,t)$ evaluated in the direct numerical simulation (DNS) coincides almost perfectly with that predicted by the three-wave variant of Hasselmann's theory already in the very initial stage of evolution such as only a few typical periods. Fig. 1 shows an example of $\varepsilon(k,t)$ obtained by the DNS and Hasselmann's theory in that study. The solid line shows $\varepsilon(k,t)$ predicted by Hasselmann's theory, and the dashed line shows $\varepsilon(k,t)$ which is estimated from $E(k,t)$ obtained by the DNS of the original deterministic equation

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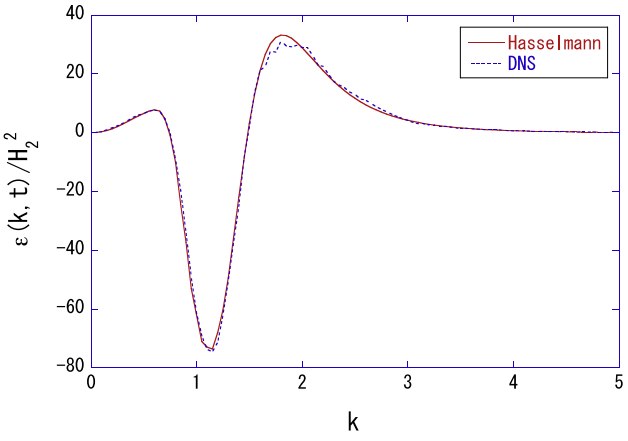

Fig. 1. $\varepsilon(k,t)$ estimated by DNS and Hasselmann's theory. $\varepsilon(k,t)$ in DNS is evaluated by $\varepsilon(k,t) = (E(k,2T_p) - E(k,T_p))/T_p$.

as $\varepsilon(k,t) = (E(k,2T_p) - E(k,T_p))/T_p$, where T_p being the period of the peak mode of the spectrum. Thus the $\varepsilon(k,t)$ of the DNS was estimated using a short interval of time in the very initial stage of the evolution such as just a few periods after the start of the evolution. Nevertheless it appears to coincide almost perfectly with that predicted by Hasselmann's theory.

This numerical result apparently contradicts the process of derivation of Hasselmann's equation where an application of the asymptotic relation:

$$\frac{e^{i\Delta\omega t}-1}{\Delta\omega} \to -\frac{\mathcal{P}}{\Delta\omega} + \pi i\delta(\Delta\omega) \quad (t\to\infty) \tag{1}$$

is indispensable. Here $\mathcal{P}$ means the Cauchy principal value, and the limit holds in the sense of generalized functions. This peculiar result has triggered the present study. The agreement of the $\varepsilon(k,t)$ obtained by the DNS with Hasselmann's theory in an unexpectedly earlier stage of the spectral evolution has also been reported previously by one of the authors [14,15].

Janssen stated in [6] that the time required for the *resonance function* $\sin(\Delta\omega t)/\Delta\omega$ to evolve toward a delta function $\delta(\Delta\omega)$ may be so large that considerable changes in $n(\boldsymbol{k},t)$ may have occurred in the meantime. However, the authors' previous result as above has made us wonder if such an interval of time would ever exist where we should use Janssen's equation instead of Hasselmann's equation.

Hasselmann's and Janssen's equations are originally derived for surface gravity waves. Their dispersion relation $\omega = \sqrt{gk}$ is of non-decay type and does not allow three-wave resonance. Then the deterministic equation for the complex amplitude $a(\boldsymbol{k},t)$ of each wave mode, which is often referred to as the Zakharov equation [16,17], contains convolution integrals of functions which are cubic with respect to $a(\boldsymbol{k},t)$. To evaluate $\varepsilon(k,t)$ in the DNS which is estimated from the slight change of $E(k,t)$ generated in a very short interval of time like a fraction of T_p, we should be able to detect that slight change of $E(k,t)$ with high accuracy. Such high accuracy demands a very high density of modes on the $\boldsymbol{k}$ plane which are involved in the DNS, hence requiring a vast amount of numerical computations.

Therefore, as a first step toward a clearer and detailed understanding on the relations among $\varepsilon(k,t)$'s predicted by Hasselmann's theory, Janssen's theory and the DNS of the original deterministic equation, we restrict our investigation here to the case of a cubic Hamiltonian system with a decay-type dispersion relation (hereinafter denoted simply as a three-wave system), for which the numerical burden is much lighter. As the three-wave system, we employ the same Hamiltonian system as that used in the authors' previous paper [12], which is briefly explained below.

There are many earlier works in which Hasselmann's equation and Janssen's or other types of generalized kinetic equation are compared in the context of the prediction of ocean wave spectrum. (See, for example, [8,11].) The main concern of these works is the difference between the spectra predicted by different kinetic equations, and the relevant time-scale is typically $1000T_p$. On the other hand, we are mainly interested here not in the long-term prediction of $E(k,t)$ but in the detailed behavior of the rate of change $\varepsilon(k,t)$ of $E(k,t)$ in the initial stage of the spectral evolution, and the typical time-scale in this work is the linear time-scale comparable to T_p. We will show that the $\varepsilon(k,t)$ given by Janssen's equation changes rapidly in this linear time-scale. Incidentally, we use the term *initial stage* in the sense that the phases of wave modes with different $\boldsymbol{k}$ are assumed to be random and uncorrelated at $t=0$, and we treat the range of t such as $0 < t < 50T_p$. Therefore the term initial stage here can be rephrased as *shortly after phase mixing* if we follow the usage of term in [8,11].

The problems which will be addressed in the paper include: (i) how large the difference between the $\varepsilon(k,t)$ given by Hasselmann's theory and that given by Janssen's theory is, (ii) how long time should be elapsed for Janssen's $\varepsilon(k,t)$ to give practically the same value as Hasselmann's $\varepsilon(k,t)$, and how the time scale depends on the wavenumber k, (iii) how Janssen's $\varepsilon(k,t)$ approaches to Hasselmann's $\varepsilon(k,t)$, whether it approaches monotonically or with a damped oscillation, and so on.

This paper is organized as follows. The theoretical background of the study is presented in Section 2. Numerical method to evaluate the rate of change of the spectrum based on the DNS, Hasselmann's equation, and Janssen's equation are also explained there. In Section 3, the numerical results of DNS, Janssen's equation, and Hasselmann's equation are compared with each other. In Section 4, we investigate the asymptotic behavior of $\varepsilon(k,t)$ estimated by Janssen's theory which is reported in Section 3 in more detail and give a reasonable explanation to it. The final Section 5 summarizes the study.

2. Problem setting

2.1. Theoretical background

In order to quantitatively investigate the relations between $\varepsilon(k,t)$'s given by Hasselmann's theory, Janssen's theory, and the DNS, we employ a three-wave Hamiltonian system which is defined as follows. This system is exactly the same as that used in the authors' previous paper [12]. The Hamiltonian of the system is defined as

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3, \tag{2a}$$

$$\mathcal{H}_2 = \int \omega(\boldsymbol{k})|a(\boldsymbol{k})|^2 d\boldsymbol{k}, \tag{2b}$$

$$\mathcal{H}_3 = \frac{1}{2}\int \left\{V(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)a^*(\boldsymbol{k})a(\boldsymbol{k}_1)a(\boldsymbol{k}_2) + \text{c.c.}\right\} \times \delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)d\boldsymbol{k}_{123}. \tag{2c}$$

$$\omega(\boldsymbol{k}) = k^{3/2}, \quad V(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2) = (kk_1k_2)^{1/4}, \tag{2d}$$

and the corresponding canonical equation reads

$$\begin{aligned}\frac{da(\boldsymbol{k})}{dt} &= -i\frac{\delta\mathcal{H}}{\delta a^*(\boldsymbol{k})}\\ &= -i\omega(\boldsymbol{k})a(\boldsymbol{k})\\ &\quad - \frac{i}{2}\int V(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)a(\boldsymbol{k}_1)a(\boldsymbol{k}_2)\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)d\boldsymbol{k}_{12}\\ &\quad - i\int V^*(\boldsymbol{k}_1,\boldsymbol{k},\boldsymbol{k}_2)a(\boldsymbol{k}_1)a^*(\boldsymbol{k}_2)\delta(\boldsymbol{k}_1-\boldsymbol{k}-\boldsymbol{k}_2)d\boldsymbol{k}_{12}.\end{aligned} \tag{3}$$