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Resonant interfacial capillary–gravity waves in the presence of damping effects

M.C.W. Jones

Department of Mathematics, Kingston University, Kingston-upon-Thames KT1 2EE, UK

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ABSTRACT

An investigation is made into the waves of small and moderate amplitude which may occur at the interface of two inviscid fluids of different densities. The external forces are those of gravity and surface tension and the waves are due to the resonant interaction between the Mth and Nth harmonics of the fundamental mode. In contrast to previous studies, damping effects are taken into account. Important parameters in the problem are the velocity and density ratios. A pair of coupled nonlinear Schrodinger-type partial differential equations for the wave amplitudes is derived which model the evolution of the waves, correct up to third order. A wide variety of sinusoidal solutions to the equations is shown to exist, irrespective of the values assigned to the parameters. The stability of these solutions to small modulational perturbations is considered. It is found that when the damping is due to dissipation then the waves are stabilised.

1. Introduction

In this paper we continue an investigation into the evolution, nature and stability of the resonant capillary–gravity waves which occur on the interface of two ideal fluids, each of semi-infinite vertical extent. One of the earliest pieces of work on resonant water-waves was that of Wilton [1]. He considered the waves which occur on the free surface of a fluid and arise due to the interaction between the fundamental mode and its second harmonic. These waves now bear his name. The topic then, perhaps surprisingly, lay relatively dormant until the 1960's when Pierson and Fife [2] reconsidered the Wilton ripple phenomenon by employing the method of multiple scales to derive some power series expansions of the possible wave profiles. In a series of papers Nayfeh used similar methods to study both second and third harmonic resonances. In Nayfeh [3–5] he looked at second and third harmonic resonant waves in the cases of both deep water and finite depth while in [6] he studied second harmonic resonances between an air stream and a train of capillary–gravity waves. At around the same time McGoldrick [7,8] published a pair of papers which used similar techniques to study second harmonic resonances and also considered briefly the more general interaction between the fundamental and higher order modes. The first study of perfectly general *M*-*N* resonances appears to be that of Chen & Saffman [9] who employed weakly nonlinear methods to give a fairly complete description of the kinds of waves which may occur on a free surface. Later Jones & Toland [10] cast the problem as one in bifurcation theory and used methods of functional analysis to obtain results which largely confirmed Chen & Saffman's conclusions, a similar analysis was conducted by Okamoto [11]. Subsequently Jones went on to consider the resonant waves which occur at the interface of two fluids. In [12] and [13] he regarded the problem as one in bifurcation theory and deduced results concerning the existence and multiplicity of solutions. In [14] he employed the method of multiple scales to derive a pair of coupled nonlinear Schrodinger equations which describe the evolution of the interface and presented sinusoidal-type solutions to these equations whose stability was analysed.

The stability of wavetrains has a long history but one of the major results of the last fifty years is that of Benjamin & Feir [15] who showed that a nearly monochromatic gravity plane wavetrain of moderate amplitude on deep water is unstable in the presence of small sideband perturbations. Shortly afterwards Zakharov [16] showed how the evolution of such a wavetrain may be described by the nonlinear Schrodinger equation and how this may be used to deduce similar results concerning its stability. In a very interesting paper [17], Segur and his co-workers conducted a careful re-appraisal of the conclusions of Benjamin & Feir. They examined uniform trains consisting of capillary–gravity waves and took into account the effects of dissipation. Their results showed that the Benjamin & Feir instability may be stabilised by any finite amount of dissipation, however small, see also [18].

Other noteworthy studies of interfacial capillary–gravity waves are to be found in [19–21]. In [19] a study is made of the waves which arise between fluids of finite and unequal depths. Initially a variational approach is employed in order to derive the Lagrangian formulation of the problem. Both travelling waves, standing waves

E-mail address: mark.jones@kingston.ac.uk.

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and mixed waves (i.e. those which are formed by an interaction between standing and travelling waves) are considered. Subsequently the authors employ the method of multiple scales by means of which they derive an equation of the Davey–Stewartson type in order to analyse the stability of the waves. The topic of [20] is a study of the resonant waves which are formed at the interface of two fluids of infinite vertical extent. It deals with the case which arises when the second harmonic and the fundamental mode are at near resonance and also employs a Lagrangian formulation to derive the normal form of the problem. However, this particular resonance is specifically excluded from this report. A more geometrical approach was taken in [21] which exploits the symmetries inherent in the problem and derives a new Hamiltonian formulation. Most of this work is devoted to non-resonant waves but a singularity leading to nonlinear resonance is identified and investigated numerically.

In this paper we extend these studies and investigate the resonant interfacial capillary–gravity waves which are formed by the interaction of the Mth and Nth harmonics of the fundamental wave where $M > N$, (for technical reasons the cases $M = 2N$ and $M = 3N$ are excluded). It is shown how the evolution of these interfaces may be modelled by a pair of coupled nonlinear partial differential equations of a nonlinear Schrodinger type. These generalise the equations appearing contained in [14] which considered the same problem at precise resonance. In this report imperfections or damping effects are taken into account. These imperfections can be caused in a variety of ways. As considered in [22], they could be as a result of dissipation, or since we are considering resonant waves they could be as a result of waves which are close to, but not equal to, the critical wavelength.

A very wide class of solutions to these equations are exhibited, generalising the somewhat restricted set given in [14]. In contrast with that report, no restrictions are imposed on the values of the parameters present in the problem. We then proceed to consider the stability of the interfaces. In the case when the resonance is exact many differing stability portraits were found to be possible, depending on the chosen values of the parameters. However, when dissipation is present the waves are always stable. The presence of damping in general and dissipation in particular would hence seem to be very significant and important in the study of water waves in nature. For instance in [23], Snodgrass et al. successfully tracked ocean swell across the whole expanse of the Pacific ocean, whereas according to the nondissipative theory of [15] and [16], such waves are intrinsically unstable.

2. Setting the scene

We shall be concerned with the irrotational motion of two inviscid and incompressible fluids, each of infinite horizontal and vertical extent. We shall choose a three dimensional Cartesian coordinate system so that when the system is in its unexcited state the interface of the fluids is given by $z = 0$, the lower fluid is moving everywhere with constant velocity V_1 in the x-direction and the upper fluid is moving everywhere with constant velocity V_2 in the x-direction. (Note that we do not assume the $V_i's$ to be positive so the flows need not be uni-directional.) When the system is in its disturbed state the interface is given by $z = H(x, y, t)$. Throughout the paper the subscript 1 is used to denote quantities associated with the lower fluid (that occupying $z \le 0$) and 2 is used to denote those associated with the upper fluid (that occupying $z \ge 0$). We shall denote the densities of the fluids by ρ_i where $\rho_1 > \rho_2$ so that the heavier fluid is the lower. We define the relative density ρ to be ρ_2/ρ_1 which clearly lies between zero and unity. Since the motion is irrotational we may introduce the velocity potentials for the sinusoidal disturbances $\varphi_i(x, y, z, t)$ in each fluid. The forces on the fluid are gravity g which acts in the negative z direction and the surface tension S which acts at the interface.

We shall be interested in the motion arising from the interaction of the Mth and Nth modes of the fundamental where M and N are fixed but arbitrary integers. To this end we introduce the notation $E(n) = \exp in(x - \omega t)$, where n is any integer, ω is the fundamental frequency and we have normalised the wavenumber to unity. We also introduce a small positive parameter ε which acts as a measure of the interface steepness and the 'slow variables' $X = \varepsilon x, Y = \varepsilon y, T_0 = \varepsilon t$ together with the 'very slow variable' $T = \varepsilon^2 t$. The notation T_0 might look slightly ugly but it turns out that T_0 drops out of the analysis at a fairly early stage and we work mainly with T, X and Y. The method of multiple scales is now employed in order to derive the governing equations of the motion. For more detailed descriptions of this procedure, see [14,24–31].

The equations governing the motion are then

$$\nabla^2\varphi_1 = 0, \quad z \le H, \tag{1}$$

$$\nabla^2\varphi_2 = 0, \quad z \ge H, \tag{2}$$

$$\nabla\varphi_1 \to 0, \quad z \to -\infty, \tag{3}$$

$$\nabla\varphi_2 \to 0, \quad z \to \infty, \tag{4}$$

$$H_t - \varphi_{jz} + (V_j + \omega)H_x + \varphi_{jx}H_x + \varphi_{jy}H_y = 0, \quad z = H \quad j = 1, 2 \tag{5}$$

$$\begin{aligned}
&\rho\varphi_{2t} - \varphi_{1t} + \rho(V_2 + \omega)\varphi_{2x} - (V_1 + \omega)\varphi_{1x} - \frac{MN}{M+N}(\rho V_2^2 + V_1^2)H \\
&+ \frac{\rho}{2}(\varphi_{2x}^2 + \varphi_{2y}^2 + \varphi_{2z}^2) - \frac{1}{2}(\varphi_{1x}^2 + \varphi_{1y}^2 + \varphi_{1z}^2) \\
&+ \frac{(\rho V_2^2 + V_1^2)(H_{xx}(1 + H_y^2) + H_{yy}(1 + H_x^2) - 2H_xH_yH_{xy})}{(M+N)(1 + H_x^2 + H_y^2)^{3/2}} \\
&+ \rho\varepsilon^2\delta\,\varphi_2 - \varepsilon^2\delta\,\varphi_1 = 0, \quad z = H.
\end{aligned} \tag{6}$$

In the governing equations the coefficients in the boundary conditions (5) and (6) have been chosen to ensure that the linearised forms are satisfied by the Nth and Mth harmonics. Details may be found in [14,27,29].

In these equations, the parameter δ represents damping effects. This damping can be caused by a variety of mechanisms, for instance slight variations in the frequency of the wave maker; the presence of cross waves or the Raman effect. In [32] Miles considered damping due to dissipation. In this case δ is positive and he developed analytic forms for δ based on the various types of dissipation which can occur in deep water systems. It is of course possible to perturb the equations in many other ways to account for damping effects. For instance Dias et al. [33] present a formulation in which both Bernoulli's and the kinematic boundary conditions are perturbed and justify it rigorously for free surface gravity waves. The work of Zhang & Vinals [34] included a perturbation term of the form $\delta\phi_{zz}$ while in [3] Nayfeh perturbed the wave number. In [35,36] Jones perturbed the term involving H_{xx} in (6) while in [29] he replaced the leading order term C_N in the expansion of H (see (9) below) with $C_N e^{-i\varepsilon\,\delta\,t}$ and similarly for C_M.

The solutions $\phi = 0$ and $H = 0$ to the system ((1)–(6)) represent horizontal laminar flow with velocities V_1 and V_2 in the x-direction in the lower and upper fluids respectively. The interface is given by $z = 0$.

The next step in the analysis is to expand the velocity potentials and the interface profile in ascending powers of ε in order to take into account the disturbance due to the two harmonics. Up to appropriate order, the relevant expansions are

$$\begin{aligned}
\varphi_1 = &[\ \varepsilon iV_1C_N + \varepsilon^2(A_N^{(2)} + zV_1C_{NX}) \\
&+ \varepsilon^3(A_N^{(3)} - izA_{NX}^{(2)} - \frac{iz^2}{2}V_1C_{NXX} - \frac{iz}{2}V_1C_{NYY})\ ]\,E(N)e^{Nz} \\
&+ [\ \varepsilon iV_1C_M + \varepsilon^2(A_M^{(2)} + zV_1C_{MX})
\end{aligned}$$