European Journal of Mechanics / B Fluids 68 (2018) 100–107

Contents lists available at ScienceDirect

European Journal of Mechanics / B Fluids

journal homepage: www.elsevier.com/locate/ejmflu

Current sheets as waveguides for the fast magnetosonic wave

Manuel Núñez

Depto. Análisis Matemático, Universidad de Valladolid, 47011 Valladolid, Spain

ARTICLE INFO

Article history:
Received 26 April 2017
Received in revised form 31 October 2017
Accepted 28 November 2017
Available online 5 December 2017

Keywords:
Current sheets
Fast magnetosonic waves
Nonlinear geometric optics

ABSTRACT

The geometry of rays and wavefronts associated to the fast magnetosonic wave is studied for a planar cold plasma, in an equilibrium where magnetic null points and current sheets are present. It is found (a) that null points attract rays and singular ones repel them, causing a characteristic evolution of wavefronts; and (b) that current sheets act as waveguides, directing both rays and wavefronts towards them. High frequency fast waves remain continuous when crossing the sheet; depending on the initial condition, these waves may become shocks before or after the sheet.

1. Introduction

The connection between magnetohydrodynamic waves and magnetic reconnection is well attested in several astrophysical settings, notably in the solar corona [1]. One of the main features of classical models of fast magnetic reconnection is the presence of magnetic null points [2–4]: thus the study of MHD waves in equilibria possessing magnetic nulls is worth pursuing. [5] is an excellent review, and specific applications to solar reconnection may be found e.g. in [6–8]. In many of these papers the plasma is assumed cold, or to have a low beta, meaning that the sound speed is so much smaller than the Alfvén one that it may be ignored. In this case no MHD wave can travel over a magnetic null, since the fast magnetosonic speed is $B/\sqrt{\rho}$, the Alfvén wave speed is $(\mathbf{B}\cdot\mathbf{n})/\sqrt{\rho}$, and the slow wave speed is zero; $\mathbf{B}$ represents the magnetic field, ρ the density and $\mathbf{n}$ the wave vector. Thus magnetic null points behave like rocks around which MHD waves may curl and break, but never surpass. Geometric optics (also called geometric acoustics) have been applied successfully to null points in the solar atmosphere, highly relevant in the study of active regions. In [9], the rays for the single null case are found both for cold and warm plasmas, and both caustics and shock waves are described; whereas in [10], slow magnetosonic waves in solar active regions are studied. The relation between isolated magnetic nulls and formation of fast shocks is also analyzed in [11] by geometric optics methods. While fast waves are attracted by null magnetic points, the slow ones do not reach those points (where reconnection occurs) with any intensity. However, it appears that these may generate fast waves due to MHD mode transmission, in particular at points where the Alfvén and sound speeds coincide. Nevertheless, this is not the only relevant configuration: several geometries involve a whole line where the magnetic field changes direction, and therefore a singular current appears. This occurs for the classical model of Sweet and Parker for slow reconnection [12–14], as well as in the models of fast reconnection of Petschek, Axford and Sonnerup [15–18], and, more to the point, in the wide sheet model pioneered by Syrovatskii [19,20]; see also [21,22]. All these models have been refined, extended and argued about to account for the complexities of real world magnetic reconnection, but independently of particular preferences the study of fast magnetosonic waves in equilibria possessing current sheets is interesting in itself. We will use the methods of geometric optics to find the evolution of rays and wavefronts. The basics may be found in [23–25], while applications to MHD and shock formation occur in [26,27]. Details on the MHD equations, eigenvalues and eigenvectors are well explained in [28]. We will analyze the rays and wavefronts starting from a single front in progressively more complex geometries: first with a single critical X-point of the magnetic field; then with two X-points connected by an O-point. This is the correct topology when e.g. a change in boundary conditions splits an isolated X-point, but from the physical viewpoint is unnatural, since the size of the magnetic field at the O-point is infinite. The truth is that a current line connecting the two X-points forms and the magnetic field becomes discontinuous: this is our third configuration.

2. Equilibrium, rays and wavefronts

For any static equilibrium of the ideal MHD equations the Lorentz force must be compensated by the kinetic pressure. In normalized units:

$$(\nabla\times\mathbf{B})\times\mathbf{B}=\nabla P. \tag{1}$$

E-mail address: mnjmhd@am.uva.es.

https://doi.org/10.1016/j.euromechflu.2017.11.009
0997-7546/

A state equation for the pressure must be added. For an isothermal or polytropic plasma we have $P = P(\rho)$, ρ being the fluid density. The sound speed c_s satisfies $c_s^2 = \partial P/\partial\rho$; the Alfvén speed c_A satisfies $c_A^2 = B^2/\rho$, where $B = |\mathbf{B}|$. Since the assumption $P = P(\rho)$ means $\nabla P = c_s^2\nabla\rho$, if we assume a low beta or cold plasma (i.e. that sound speed is very small as compared with the Alfvén one), we conclude $\nabla P \sim \mathbf{0}$, which agrees well with the hypothesis $\nabla P = \mathbf{0}$ in Syrovatskii's model [3]. The fluid is in general compressible. Indeed, for large times and a current sheet of constant width, the plasma tends to rarify near the sheet [20]. However, since for large times the model is doubtful anyhow (see e.g. [29]), we admit the quasi-static character of the model and consider only the initial stages of the evolution, when one starts with a constant density. Since the spatial gradients of the magnetic field are then much larger than the ones of the density, we assume that $\rho \sim$ const. (Taken to 1 for convenience). Therefore the Alfvén velocity equals B.

For a planar plasma $\nabla \times \mathbf{B}$ is orthogonal to $\mathbf{B}$, so necessarily $\nabla \times \mathbf{B} = \mathbf{0}$ and the plasma is current-free (except in the regions of the domain where it is singular). That means that if $\mathbf{B} = (B_x, B_y)$, those functions satisfy

$$\frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x}$$
$$\frac{\partial B_x}{\partial x} = -\frac{\partial B_y}{\partial y}, \tag{2}$$

the last equation because of $\nabla\cdot\mathbf{B} = 0$. In other words, calling $w = B_y + iB_x$, w satisfies the Cauchy–Riemann equations and therefore is an analytic function. We will analyze the fast waves associated to the certain well known configurations by the methods of geometric optics. For this to be a valid approximation, the wavelength of the perturbed wave must be small compared to the typical lengths of the equilibrium: equilibria with rapid spatial variation are not appropriate for this method. Our first equilibrium, involving a single null point of the magnetic field, varies smoothly and therefore is not suspect. However, our second equilibrium includes a singular point of the field and the third one a discontinuity, so that in principle geometric optics cannot be applied. The reasons why this approach is nonetheless acceptable are explained in the respective cases.

Let us remember the basics of geometric optics. Consider a quasilinear hyperbolic system:

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_j A_j(\mathbf{x}, \mathbf{u})\frac{\partial \mathbf{u}}{\partial x_j} + \mathbf{C}(\mathbf{x}, \mathbf{u}) = \mathbf{0}. \tag{3}$$

For any spatial vector $\mathbf{k}$ and equilibrium state $\mathbf{u}_0$ take a fixed eigenvalue $\Lambda(\mathbf{k})$,

$$\det(\Lambda(\mathbf{k})I + A_j(\mathbf{x}, \mathbf{u}_0)k_j) = 0. \tag{4}$$

The eikonal equation associated to this eigenvalue is

$$\frac{\partial\phi}{\partial t} = \Lambda(\nabla\phi), \tag{5}$$

and ϕ is the phase. In our case the system will be the ideal MHD one, and we choose for Λ the fast magnetosonic frequency (see e.g. [28]). If $\mathbf{u}_0$ corresponds to a static state with pressure P, density ρ and magnetic field $\mathbf{B}$,

$$\Lambda(\mathbf{k})^2 = \frac{1}{2}\left(\frac{\partial P}{\partial\rho} + \frac{B^2}{\rho}\right)|\mathbf{k}|^2 + \frac{1}{2}\left[\left(\frac{\partial P}{\partial\rho} + \frac{B^2}{\rho}\right)^2|\mathbf{k}|^4 - 4\frac{\partial P}{\partial\rho}\frac{(\mathbf{B}\cdot\mathbf{k})^2}{\rho}|\mathbf{k}|^2\right]^{1/2}. \tag{6}$$

Rays are solutions of the system

$$\frac{d\mathbf{x}}{dt} = \nabla_{\mathbf{k}}\Lambda(\mathbf{x}, \mathbf{k})$$
$$\frac{d\mathbf{k}}{dt} = -\nabla_{\mathbf{x}}\Lambda(\mathbf{x}, \mathbf{k}). \tag{7}$$

The phase is constant along rays,

$$\frac{d}{dt}(\phi(t, \mathbf{x}(t))) = 0. \tag{8}$$

Often one takes a normalized vector $\mathbf{n} = \mathbf{k}/|\mathbf{k}|$ and uses the frequency

$$c(\mathbf{n}) = \frac{\Lambda(\mathbf{k})}{|\mathbf{k}|}. \tag{9}$$

Eqs. (7) for the plane may be written in terms of c, $\mathbf{n}$ and its orthogonal $\mathbf{n}^\perp$, chosen so that $\{\mathbf{n}, \mathbf{n}^\perp\}$ form an orthonormal positively oriented system:

$$\frac{d\mathbf{x}}{dt} = c\mathbf{n} + (\mathbf{n}^\perp\cdot\nabla_{\mathbf{n}}c)\mathbf{n}^\perp, \tag{10}$$
$$\frac{d\mathbf{n}}{dt} = -(\mathbf{n}^\perp\cdot\nabla_{\mathbf{x}}c)\mathbf{n}^\perp. \tag{11}$$

For static equilibria, and in terms of the speed of sound, Alfvén speed, and the angle χ that forms the magnetic field $\mathbf{B}$ with $\mathbf{n}$, the fast magnetosonic frequency $c(\mathbf{n})$ satisfies:

$$c(\mathbf{n})^2 = \frac{1}{2}(c_s^2 + c_A^2) + \frac{1}{2}\left[(c_s^2 + c_A^2)^2 - 4c_s^2c_A^2\cos^2\chi\right]^{1/2}. \tag{12}$$

With our hypothesis of low beta, this equation simplifies enormously:

$$c = c_A = B. \tag{13}$$

Writing

$$\mathbf{n} = (\cos\psi, \sin\psi)$$
$$\mathbf{n}^\perp = (-\sin\psi, \cos\psi), \tag{14}$$

the ray equations (11) plus identity (13) yield

$$\frac{dx}{dt} = B\cos\psi \tag{15}$$
$$\frac{dy}{dt} = B\sin\psi \tag{16}$$
$$\frac{d\psi}{dt} = \frac{\partial B}{\partial x}\sin\psi - \frac{\partial B}{\partial y}\cos\psi. \tag{17}$$

This implies

$$\frac{d\ln B}{dt} = \frac{\partial B}{\partial x}\cos\psi + \frac{\partial B}{\partial y}\sin\psi. \tag{18}$$

Let arg be one of the arguments of the analytic function w, chosen so as to be continuous along the rays under study. Recall that if the domain of w is not simply connected because of the presence of a singularity or a current sheet, arg w is a multiform function and its value may change as one passes twice over the same point of the domain; this, however, will not be an issue in the cases we study. Since $\log w = \ln B + i\arg w$ is locally an analytic function, by the Cauchy–Riemann equations and (15)–(16) one finds

$$\frac{d}{dt}(\psi - \arg w) = 0. \tag{19}$$

This means that $\psi - \arg w$ is constant along the ray. If we take the start of this at $t = 0$, and calling $\theta = \arg w$, $k = \psi(0) - \theta(0)$, Eqs. (15) and (16) may be written as

$$\frac{dx}{dt} = B\cos(\theta + k)$$
$$\frac{dy}{dt} = B\sin(\theta + k). \tag{20}$$