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A numerical continuation approach for computing water waves of large wave height

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ABSTRACT

We analyse an algorithm for the calculation of travelling water waves in flows with constant and variable vorticity. The algorithm is based on numerical continuation techniques, which are suitably adapted to the water wave problem. Numerical examples illustrate the performance of the algorithm for flows of constant vorticity, where the results are compared with the literature. We observe agreement with already existing results, but we also have some new qualitative and quantitative results considering the characteristics of the water waves both for constant and variable vorticity.

1. Introduction

We are studying two-dimensional waves travelling at constant speed c, for different vorticities. This problem has practical interest because it is related with the detection of non-uniform underlying currents from the surface wave pattern. The characterization 'travelling' means that in a two-dimensional frame moving with the constant speed c, the flow pattern – and, in particular, the shape of the surface of the fluid – does not change over time. Two-dimensionality means that the waves propagate in a fixed horizontal direction and the flow presents no variation in the horizontal direction orthogonal to the direction of wave propagation. For this reason, it suffices to analyse a vertical cross-section of the flow, parallel to the direction of wave propagation. To model sea waves of large amplitude the assumptions of inviscid flow in a fluid of constant density are appropriate and the effects of surface tension are negligible—see the discussion in [1].

It is worth highlighting the physical importance of flows with vorticity in modelling wave–current interactions; general discussions of the physical implications of rotational flow may be found in the references [1] and [2]. Recent studies have pointed out that periodic travelling waves which propagate at the surface of water with a flat bed in a flow of constant vorticity must be symmetric if no flow-reversal occurs and if the wave profile is monotone between successive crests and troughs, see [3,4]. This means that an underlying non-uniform current of constant vorticity does not break the symmetry of irrotational wave trains. Thus, following this formulation, we make the assumption of no flow-reversal in this work: In the absence of flow reversal, the approach developed in [5] ensures the real-analyticity of all the streamlines; see also the discussion in the survey [6]. In contrast to this, the available results on regularity for flow reversal (case in which critical layers appear) are merely of class C^∞, see the results in [7].

Prior to the establishment of any analytically rigorous mathematical results, numerical investigations for large amplitude waves with vorticity, were initiated by the seminal papers [8] and [9] for the infinite, and finite, depth cases respectively. Indeed, due to significant technical complications, rigorous mathematical analysis establishing the existence of large amplitude water waves has only recently been achieved. In the case of irrotational waves this was achieved by the work of Toland and collaborators (where [10,11] represent nice surveys of this topic), and for waves with vorticity, following the seminal paper [12], the existence of large amplitude waves was established for various physical generalizations in [13–16].

In this work, we follow the analytical formulation derived in [12], and reviewed in several other works, see for example [17,18]. In this approach, the curve of solutions is extensively analysed and a crucial ingredient is the existence of a branch of the bifurcating diagram, which contains flows beneath genuine waves.[1] In particular, this branch starts from a special laminar solution, the existence of which is connected with the so-called

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[1] In the sense that they are not of zero amplitude.

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dispersion relation; this curve may be continued, using arguments from bifurcation theory, to a global continuum which contains at the limit a flow with a stagnation point. The characteristic of the latter wave is that its horizontal velocity approaches the constant speed of propagation c, being related to waves of maximal amplitude, hence providing the analogue to the Stokes' extreme wave. Computations which are based on this approach are performed in [19,20], with several interesting results which agree with the relevant analytical predictions. Among the most important results of those works we point out the following: The stagnation can occur, not only at the crest, but also at the point on the bottom directly below the crest. Along the bifurcation curve, for fixed relative mass flux[2] p_0, the amplitude of the wave is increasing, the depth d varies only slightly and the hydraulic head[3] Q has (in general) one turning point. Furthermore, the waves of maximal amplitude are obtained at the end of the bifurcation curve and the maximal amplitude is an increasing function of $|p_0|$, in the case of constant vorticity. Finally, the shapes of the streamlines of the extreme waves depend on the vorticity, which is a result observed also in several numerical works and indicates the importance of the effect that the vorticity has on the features of water waves.

In our work, we analyse a numerical continuation approach tailor-made for the above described mathematical formulation, enriched with techniques appropriate to overcome some particular obstacles of this problem (turning points, bifurcation points and stagnation points). The idea on reconstructing the bifurcation branch described in [12], via numerical continuation techniques, was introduced in [19]. In our algorithm, among other techniques which are described in more detail in Section 4, we make explicit usage of analytical results obtained in [12]—instead of evaluating numerically the relevant features, through the solution of an eigenvalue problem. In more detail:

- The explicit position of the bifurcation point, which is the starting point of our iterative numerical scheme is obtained analytically through [12,18] and [21] for different cases of vorticity.
- Starting from the above point, the favourable direction for the numerical continuation, which will lead to the reconstruction of the branch containing the non-laminar flows (as opposed to the trivial one), is obtained analytically in [12,22] and [21], for the different cases of vorticity.
- The careful refining of the mesh described in Section 4 allows for the computation of waves closer to stagnation than the ones obtained in [19].
- The adaptation (described in Section 3) of the numerical continuation technique on this formulation of the water wave problem allows the systematic study of the main features of the wave and the improvement of the algorithm; as examples we note the adaptation of the step-size in the 'predictor stage' of our method and the interchange of the continuation parameter.

This has as a result an efficient and inexpensive algorithm, which, additionally, gives rise to new parts of the interesting branch of the bifurcation diagram, i.e. to families of waves with novel characteristics, see for example Section 4.2. Furthermore, we find agreement with already existing numerical results while observing additional characteristics close to stagnation, which in some cases is important enough to give an additional understanding on the solutions of the problem. Finally, in order to show the generality of our algorithm, we compute some cases of continuous and non-constant vorticity; this results in, also, an interesting family of water waves, which is depicted in Figs. 23–24.

We find worth-mentioning a series of computational works, for example [23–27], which are based on a different mathematical formulation of the problem. In particular, they are based on the analysis of [28,29]; see also [30,31] and [32] for extending the latter work in a domain with fixed and moving bottom, respectively. One can observe that our computational results are in qualitative agreement with results of the above-mentioned works.

In the recent work [33] an extensive numerical study of periodic travelling water waves is made. This work is based on a conformal mapping which is described also in [34,35] and their results are obtained through a combination of analytical techniques and spectral methods. This approach has the advantage over our work that the computation of waves with stagnation points (both on the boundary and in the interior) are possible. On the other hand this approach treats only the case of constant vorticity; one could find potential on extending this work to general vorticity using the analysis of [36] and [37]. Our approach, although having the disadvantage on stagnation points, provides results for more general distributions of vorticity; as an example we provide some examples in Section 4.3. Moreover, our computation reveals some interesting behaviour of the pressure for a particular family of waves which is discussed in Section 4.2 and it is novel, up to our knowledge. Even though that for waves that do not have stagnation points there is qualitative agreement between our work and the above-mentioned one, the most interesting cases of that work involves waves with internal stagnation (which we do not compute). Finally, the rigorous and *exact* correspondence of these two formulations – the conformal mapping used in [33] and the semi-hodograph transformation used in [12] and here – is still an open question.

The paper is organized as follows: In Section 2, we make a brief review of the mathematical formulation of the problem. In Section 3, we present the numerical continuation method, which we employ in order to compute the water waves along the bifurcating curves. In Section 4, we present the numerical results of the above procedure and we discuss the special characteristics of the waves, depending on the different values of constant vorticity. Moreover, in order to show the generality of our algorithm, we present the relevant results for some case that the vorticity is not constant; in these examples the vorticity varies linearly, as well as quadratically, with respect to the stream function. In particular, we illustrate the wave profiles, as well as other characteristics such as the velocity profile and the pressure throughout the fluid. The knowledge of these flow characteristics is very useful in qualitative studies, see [38,39]. Moreover, in practice information on the state of the sea surface is often gathered from subsurface pressure and/or velocity measurements; we refer to the discussions in [13,26,40–43].

2. The basic equations and terminology

In this section we present the governing equations for periodic two-dimensional travelling water waves in flows of constant and variable vorticity over a flat bed.

Let us denote the height function of the wave above the flat bottom by $h(q, p)$. This function satisfies the nonlinear boundary value problem, defined by (1)–(3). We omit here the derivation of (1)–(3) from the physical problem which was described in the Introduction. Instead, for the derivation of the constitutive equations (1)–(3) and the mathematical formulation of the problem we refer to [12,17] for a detailed construction, to [18,19] for an extended review and to [22,44] for a brief review.

[2] The flux p_0 is defined in (9).

[3] This quantity, being indicative of the total mechanical energy of the wave, is defined in (12).