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On the recovery of traveling water waves with vorticity from the pressure at the bed

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ABSTRACT

We propose higher-order approximation formulae recovering the surface elevation from the pressure at the bed and the background shear flow, for small-amplitude Stokes and solitary water waves. They offer improvements over the pressure transfer function and the hydrostatic approximation. The formulae compare reasonably well with the asymptotic approximations of the exact relation between the pressure at the bed and the surface wave in the zero vorticity case, but they incorporate the effects of vorticity through the solutions of the Rayleigh equation. Several examples are discussed.

1. Introduction

A basic problem in oceanography is to determine wave parameters – significant wave height, significant wave period, spectral peaks, etc. – from ocean measurements. For instance, the task of tracking the genesis and propagation of tsunamis is of obvious importance. One main source of data extensively used for the purpose is pressure transducers seeded throughout the Pacific and Indian Ocean. They collect pressure readings at various water depths and transmit to monitoring stations.

This motivates an interesting mathematical question. Suppose that a wave runs in a channel of water over a long distance practically at a constant velocity without change of form, and that the value of the pressure at the bed is given, and perhaps some other information about the upstream and downstream flow. From such data, can one recover the surface elevation? Incidentally, traveling waves may be used as a means to understand more general wave motions; see [1,2], for instance.

Under the assumption that the fluid in the bulk is irrotational, a simple approach, which is in practice, for instance, in tsunami detection, is to take up the hydrostatic approximation (see [3,4], for instance):

$$\eta(x) = \frac{1}{g}p(x). \tag{1.1}$$

Here x denotes the spatial variable in the direction of wave propagation, η is the surface displacement from the undisturbed fluid depth h_0, say, and p is the dynamic pressure, measuring the departure from the hydrostatic pressure; g is the constant due to gravitational acceleration. Another is the pressure transfer function (see [3,4], for instance):

$$\mathcal{F}(\eta)(k) = \frac{1}{g}\cosh(kh_0)\mathcal{F}(p)(k). \tag{1.2}$$

Here and throughout,

$$\mathcal{F}(f)(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$$

stands for the Fourier transform of the function $x \mapsto f(x)$. Note that (1.2) becomes (1.1) in the limit as $h_0 \to 0$.

Laboratory experiments in [5], for instance, support that the pressure transfer function satisfactorily predicts the wave height. Furthermore, one can derive (1.2) consistently in the regime of small-amplitude Stokes waves in the case of zero vorticity; see [6], for instance. On the other hand, numerous studies raised doubts about the adequacy of using the linear theory; see [7–11], for instance. Note that the effects of nonlinearity and current are not negligible in shallow water or in the surf zone; see [10], for instance.

Remarkably, exact relations were derived in [1,12,13] between the trace of the pressure at the horizontal bed and the surface elevation for Stokes and solitary water waves. In particular, the formulae apply to large amplitude waves. They are implicit but, nevertheless, easily implemented in numerical computations, and

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the results agree to varying degrees with laboratory experiments; see [2], for instance.

The arguments strongly use that in the case of zero vorticity, one is to solve the Cauchy problem for the Laplace equation. Unfortunately, they cannot accommodate underlying shear flows and other physical aspects. We pause to remark that real flows are hardly irrotational. Rather vorticity is generated, for instance, by density stratification, the shear force of wind, currents or tidal forces, and the effects of bathymetry. At present, no exact relations are available between the pressure at the bed and the surface wave in rotational flows. Furthermore, numerical schemes approximating the exact formulae do not converge, because the Cauchy problem for an elliptic PDE is ill-posed.

Recently in [14], one of the authors elaborated (1.2) and (1.1) to permit vorticity and density stratification. Specifically, the pressure transfer function and the hydrostatic approximation were consistently derived for small-amplitude surface and interface waves over an arbitrary background flow. Unfortunately, they do not capture the effects of nonlinearity. Furthermore, the hydrostatic approximation does not sense the effects of vorticity.

Here we take matters further and propose higher-order approximation formulae recovering the surface elevation from the pressure at the bed, for small-amplitude Stokes and solitary water waves in an arbitrary shear flow. Specifically, we compute higher-order correction terms to the pressure transfer function and the hydrostatic approximation in [14]. To the best of the authors' knowledge, these are new. We carry out higher order perturbations of the governing equations, rather than relying on a less empirical approach of higher-order Stokes expansions. We sacrifice the ability to accommodate large amplitude waves, but we are able to work to an arbitrary degree of accuracy, albeit finite, when exact formulae relating the pressure at the bed and the surface wave are unavailable.

The formulae incorporate the effect of vorticity through the solutions to the Rayleigh equation, which one must in general investigate numerically. But we make an effort to discuss some examples. In the case of zero vorticity, in particular, we show that our results compare reasonably well with asymptotic approximations of the exact formulae in [1], for instance; see Examples 3.1 and 4.1. The upshot is a simple but highly computationally manageable method, which may develop into an effective numerical scheme. The practical use of the results, including numerical and experimental studies, is of future investigation.

2. Preliminaries

The water wave problem, in the simplest form, concerns the wave motion at the surface of an incompressible inviscid fluid, below a body of air and acted upon by gravity. For definiteness, we take Cartesian coordinates (x, y), where the x-axis points in the direction of wave propagation and the y-axis vertically upward. In other words, the motion is constant in one horizontal direction. The fluid at time t occupies a region in $\mathbb{R}^2$, bounded above by the *free* surface and below by the fixed horizontal bottom $y = 0$, say. Let $y = h(x; t)$ describe the fluid surface at time t, and we assume that h is a single-valued, non-negative and smooth function. In the bulk of the fluid, the velocity $(u(x, y; t), v(x, y; t))$ and the pressure $P(x, y; t)$ satisfy the Euler equations for an incompressible fluid:

$$\begin{cases} u_t + uu_x + vu_y = -P_x, \\ v_t + uv_x + vv_y = -P_y - g \quad \text{in } 0 < y < h(x; t), \\ u_x + v_y = 0. \end{cases}$$

Here and throughout, a subscript denotes partial differentiation. Although an incompressible fluid such as water may have variable density, we assume for simplicity that the density $= 1$. The flow is allowed to be rotational and characterized by the vorticity:

$$\omega = v_x - u_y.$$

The kinematic and dynamic conditions at the fluid surface

$$v = h_t + uh_x \quad \text{and} \quad P = P_{atm} \quad \text{at } y = h(x; t)$$

state, respectively, that each water particle at the surface remains so for all times and that the pressure at the surface equals the atmospheric pressure P_{atm}. We assume that the air is quiescent and we neglect the effects of surface tension. The flow is required to be tangential to the bottom:

$$v = 0 \quad \text{at } y = 0.$$

It is a matter of experience that waves which are commonly seen in the ocean or a lake propagate over a long distance practically at a constant velocity without change of form, namely *traveling waves*. In other words, u, v and P are functions of $(x - ct, y)$ and h is a function of $x - ct$ for some $c > 0$, the speed of wave propagation. Under this assumption, we will go to a moving coordinate frame, changing $x - ct$ to x, whereby the time t completely disappears. The result becomes:

$$\begin{cases} (u-c)u_x + vu_y = -P_x, \\ (u-c)v_x + vv_y = -P_y - g & \text{in } 0 < y < h(x), \\ u_x + v_y = 0, \\ v = (u-c)h_x \quad \text{and} \quad P = P_{atm} & \text{at } y = h(x), \\ v = 0 & \text{at } y = 0. \end{cases} \tag{2.1}$$

Note that

$$h \equiv h_0, \qquad (u, v) = (U(y), 0) \quad \text{and} \quad P = P_{\rm atm} + g(h_0 - y) \tag{2.2}$$

make a solution of (2.1) for arbitrary $c > 0$, $h_0 > 0$ and an arbitrary $U \in C^1([0, h_0])$. Physically, it represents a shear flow, for which the velocity and the fluid surface are horizontal and the pressure is hydrostatic. The present interest is waves propagating in the x-direction over a prescribed shear flow of the form. In what follows, therefore, h_0 and U are held fixed. Note that the vorticity of (2.2) is $-U'(y)$. Here and throughout, the prime denotes ordinary differentiation.

Scaling of variables. In order to systematically characterize various approximations, we introduce

$$\begin{aligned} &\delta = \text{the long wavelength parameter} \quad \text{and} \\ &\epsilon = \text{the amplitude parameter}, \end{aligned} \tag{2.3}$$

and we define the set of scaled variables. Rather than introducing a new notation for the variables, we choose, wherever convenient, to write, for instance, $x \mapsto x/\delta$. This is to be read that x is replaced by x/δ, so that hereafter the symbol x will denote a scaled variable. With this understanding, let

$$x \mapsto x/\delta \tag{2.4}$$

and

$$\begin{aligned} &u \mapsto U + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \cdots \quad \text{and} \\ &v \mapsto \delta(\epsilon v_1 + \epsilon^2 v_2 + \epsilon^3 v_3 + \cdots). \end{aligned} \tag{2.5}$$

Moreover, we write

$$h = h_0 + \epsilon\eta_1 + \epsilon^2\eta_2 + \epsilon^3\eta_3 + \cdots \tag{2.6}$$

and

$$P = P_{atm} + g(h_0 - y) + \epsilon p_1 + \epsilon^2 p_2 + \epsilon^3 p_3 + \cdots. \tag{2.7}$$

Physically, $h - h_0$ means the surface displacement from the undisturbed fluid depth and $P - P_{atm} - g(h_0 - y)$ is the dynamic