



Oblique waves on a vertically sheared current are rotational



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ABSTRACT

In the study of surface waves in the presence of a shear current, a useful and much studied model is that in which the shear flow has constant vorticity. Recently it was shown by Constantin (2011) that a flow of constant vorticity can only permit waves travelling exactly upstream or downstream, but not at oblique angles to the current, and several proofs to the same effect have appeared thereafter. Physical waves cannot possibly adhere to such a restriction, however. We resolve the paradox by showing that an oblique plane wave propagating atop a current of constant vorticity according to the linearised Euler equation carries with it an undulating perturbation of the vorticity field, hence is not prohibited by the Constantin theorem since vorticity is not constant. The perturbation of the vorticity field is readily interpreted in a Lagrangian perspective as the wave motion gently shifting and twisting the vortex lines as the wave passes. In the special case of upstream or downstream propagation, the wave advection of vortex lines does not affect the Eulerian vorticity field, in accordance with the theorem. We conclude that the study of oblique waves on shear currents requires a formalism allowing undulating perturbations of the vorticity field, and the constant vorticity model is helpful only in certain 2D systems.

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1. Introduction

The interaction of surface waves and currents has been a topic of interest for a long time. In the presence of a depth-dependent current, the nature of surface waves can change perceptibly, a problem of great technological relevance in areas where currents with near-surface vorticity are often present, such as the near-shore region and river deltas, and in the presence of currents generated by wind or tides [1]. For example, the vorticity of the tidal current in the Columbia River mouth was reported at around 0.4 s^{-1} in the top 5 m of the water column, enough to significantly affect the dispersion of gravity waves of wavelengths up to tens of metres [2]. Waves interacting with currents could be one of the key mechanisms for the generation of giant waves [3,4], and have been considered lately both analytically and numerically for vertically sheared currents where the current itself has constant vorticity [5,6]. Wave–current interactions have also been studied in the context of wave resistance for insect biolocomotion [7,8]. For two-dimensional systems of waves and shear currents, a sizeable literature exists (see, e.g., [1,9,10] and references therein), but with very few exceptions (e.g., [11–13]) wave propagation other than directly with or against the current have not been studied. The

shear current of constant vorticity with a free surface has recently also attracted much interest in the mathematical community, e.g. [14,16–18] and further references therein.

Recently Constantin [19] proved that when the vorticity is constant for a shear current with a free surface, wave propagation must be aligned either exactly upstream or exactly downstream, i.e., the flow must be effectively two-dimensional. His work furthers that of Constantin & Kartashova [15] and additional proofs of similar results have followed [20,21]. While the mathematical argument, briefly recounted below, is indisputable, the result proved in these references appears to run counter not only to physical intuition, but also appears to make the use of vectorial Fourier analysis for linear surface waves in 3D illegal when a shear flow is present, since oblique Fourier components would be forbidden.

On the other hand, recent times have seen progress made on the fully 3D problem of linear surface waves on top of a shear current of constant vorticity, and explicit solutions to the linearised Euler equations for 3D surface waves on such a background flow have been found for initial value problems [22,23], ship waves [24,25], and waves from a submerged oscillating source [26]. In all cases the solutions rely on the ability to express wave associated surface elevation and velocity components as a 2D Fourier integral in the horizontal plane, whose kernel functions can be understood as plane waves propagating in *all* horizontal directions. In fact, with only directly upstream and downstream

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wave components, neither ship wave patterns nor ring waves from an initial disturbance or a point source are possible.

A paradox emerges, therefore, because one and the same equation of motion, the Euler equation, on the one hand necessitates Constantin's theorem of two-dimensionality, and on the other hand permits solutions which are patently three-dimensional. In the following we resolve this paradox. We show that the Euler equation permits periodic plane wave solutions propagating at an angle with the underlying shear flow, but that these waves are required to carry an associated vorticity field except in the special case where wave propagation is aligned with the shear flow. Constantin's theorem rests on an assumption that the vorticity is constant everywhere, hence there is no contradiction between the theorem and the recent 3D solutions to the Euler equations in the presence of a shear flow. These skew waves are thus, in an Eulerian sense, rotational, although a Lagrangian perspective shows how the rotationality is but a slight redistribution of the vorticity of the underlying current. We show for the oblique linear plane wave, how the vorticity perturbation may be interpreted as the wave motion gently shifting, twisting, stretching and contracting the vortex lines as it travels past.

The basic plane wave solutions for an oblique wave on a linear-profile shear flow are presented, being the building bricks from which above cited results for ship waves, ring waves and the oscillating source were constructed. We propose that the theorems, which prove the non-existence of irrotational oblique waves on a shear flow with constant and horizontal vorticity, should be understood physically in a positive sense: skew waves on such a flow must always carry a corresponding vorticity perturbation, i.e., they are themselves, in an Eulerian sense, rotational. The assumption of constant vorticity, while a tempting simplification, is not helpful for the study of three-dimensional waves on shearing currents, and may be employed in some 2D systems only.

2. Two-dimensionality of constant vorticity waves

We begin by briefly recounting the result proved by Constantin [19]. Let us assume that a wave motion appears as a perturbation of a shear flow whose vorticity is constant in time and space and horizontally oriented. The flow can have finite or infinite depth, and is presumed to be inviscid and incompressible, hence the velocity field \mathbf{U} is governed by the Euler equation of motion (a dot denotes partial derivative w.r.t. time),

$$\dot{\mathbf{U}} + (\mathbf{U} \cdot \nabla)\mathbf{U} = -(1/\rho)\nabla P - g\mathbf{e}_z \quad (1)$$

and the continuity equation $\nabla \cdot \mathbf{U} = 0$. Here the pressure field is P , g is the gravitational acceleration, and \mathbf{e}_z is the vertical unit vector. Applying the curl operator yields the vorticity equation,

$$\dot{\boldsymbol{\Omega}} + (\mathbf{U} \cdot \nabla)\boldsymbol{\Omega} = (\boldsymbol{\Omega} \cdot \nabla)\mathbf{U} \quad (2)$$

where the vorticity is related to \mathbf{U} by $\boldsymbol{\Omega} = \nabla \times \mathbf{U}$.

The key assumption now made is that the velocity field has constant vorticity $\boldsymbol{\Omega}$ in time and space, not only the original shear flow, but the perturbation due to the presence of waves as well. In this case Eq. (2) reduces to

$$(\boldsymbol{\Omega} \cdot \nabla)\mathbf{U} = 0, \quad (3)$$

i.e., the velocity field can have no variation in the direction parallel to $\boldsymbol{\Omega}$. But a wave train propagating in a general direction \mathbf{k} in the xy plane must, regardless of its shape, be associated with a velocity field which varies along its direction of propagation, hence only waves propagating either directly upstream or downstream with respect to the shear flow can exist.

In particular, the above theorem implies that a plane wave of constant vorticity, which has the form

$$\zeta(\mathbf{r}, t) \propto \exp[i\mathbf{k} \cdot \mathbf{r} - i\omega(\mathbf{k})t]$$

(the real part is understood to be taken) with ζ the surface elevation and $\mathbf{r} = (x, y)$, must have \mathbf{k} pointing either exactly upstream or downstream so that $\mathbf{k} \cdot \boldsymbol{\Omega} = 0$.

For linearised wave theory this seems to disagree with the use of a Fourier description, by which *any* surface deformation can be expressed in such a form, with the appropriate eigenvalue for $\omega(\mathbf{k})$, and where contributions from \mathbf{k} in all directions are required to describe, e.g., a localised initial surface perturbation.

Moreover such a conclusion is in discord with physical intuition. A local initial perturbation of a still water surface is a classical problem considered by Cauchy and Poisson 200 years ago [27,28], and results in ring waves propagating out in all directions, with wave fronts becoming approximately plane far from the origin. The equivalent system with a uniform (irrotational) current can be found by an appropriate Galilei transformation, and the conclusion remains the same. However, Constantin's theorem seems to indicate that if a constant vorticity is now introduced, *however small*, it would drastically change the surface waves, since only up- and downstream propagation is allowed.

While there is no doubt about the soundness of the theorem itself, it seems clear that real-life wave systems cannot possibly adhere to it. The paradox is resolved in the following.

3. Solution of the linearised Euler equation for a skew plane wave

The theorems of Constantin [19], Wahlén [21] and others assume the full velocity field including the wave motion to have constant vorticity. In systems where the wave motion can be seen as a perturbation of a constant vorticity shear flow (linear and weakly non-linear system), the assumption implies that the wave motion alone carry a constant vorticity, and since it is the nature of wave motion to vary in time and horizontal space (e.g., periodically) and to decrease with depth, the assumption then realistically means the wave motion alone be irrotational.

To elucidate the situation, let us compare with the solution obtained for an oblique wave with no such restrictions imposed. Consider the linearised Euler equation, for a shear flow which itself has uniform vorticity and is of the form

$$U_{\text{curr}}(z) = U_0 + Sz \quad (4)$$

where the undisturbed surface is at $z = 0$ and the shear flow points along the x direction. The velocity field is

$$\mathbf{U} = (U_{\text{curr}} + \hat{u}, \hat{v}, \hat{w}) \quad (5)$$

where \hat{u} , \hat{v} , \hat{w} are perturbations due to the wave field, and $P = -\rho gz + \hat{p}$ with perturbation \hat{p} . We consider solutions to linear order in perturbation quantities.

A plane wave is now presumed to travel upon the shear flow (4) at an arbitrary angle θ with the x axis, i.e., the perturbation quantities \hat{u} , \hat{v} , \hat{w} and \hat{p} are all presumed to have the form

$$[\hat{u}, \hat{v}, \hat{w}, \hat{p}](\mathbf{r}, z, t) = [u(z), v(z), w(z), p(z)]e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \quad (6)$$

where $\mathbf{r} = (x, y)$ and the wave vector is $\mathbf{k} = (k_x, k_y) = k(\cos\theta, \sin\theta)$. θ is the angle between wave propagation and shear current. The eigenvalues $\omega(\mathbf{k})$ that permit a solution are provided by the free-surface boundary conditions. The system is similar to that considered in Ref. [13] and Section IV.B.3 of Ref. [1]. The Euler and continuity equations become

$$-i\omega u + ik_x U(z)u + Sw = -ik_x p / \rho, \quad (7a)$$

$$-i\omega v + ik_x U(z)v = -ik_y p / \rho, \quad (7b)$$

$$-i\omega w + ik_x U(z)w = -p' / \rho, \quad (7c)$$

$$ik_x u + ik_y v + w' = 0 \quad (7d)$$

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