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Negative energy waves in a shear flow with a linear profile

Philippe Maïssa [a,1], Germain Rousseaux [b,1], Yury Stepanyants [c,*,1]

[a] Université de Nice-Sophia Antipolis, Laboratoire J.-A. Dieudonné, UMR CNRS-UNS 7351, Parc Valrose, 06108 Nice Cedex 02, France

[b] Institut Pprime, UPR 3346, CNRS – Université de Poitiers – ISAE ENSMA, 11 Bd Marie et Pierre Curie, BP 30179, 86962 Futuroscope, France

[c] Faculty of Health, Engineering and Sciences, University of Southern Queensland, Toowoomba, QLD, 4350, Australia

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ABSTRACT

We present a derivation of the time averaged potential and kinetic energies for small-amplitude surface waves on a shear flow with constant vorticity. The effect of surface tension is also taken into consideration. It is demonstrated that the virial theorem of the energy equipartition between the potential and kinetic components is not valid in general for waves on a shear flow. We also show that waves with a negative energy may exist in a shear flow, and we find the condition for existence of such waves.

1. Introduction

The wave energy concept is of primary significance in the investigation of various problems of wave generation, propagation and absorption in hydrodynamic shear flows. Considering linear problems, we usually deal, however, with the 'quasi-energy' of monochromatic waves [1,2]. The physical meaning of the quasi-energy and its relation to the total energy of a hydrodynamic flow, including wave-induced flows, have been considered also in [3,4]. The important point following from the quasi-energy concept is that the wave energy may change its sign and become negative in the presence of a shear flow. When the negative energy waves (NEWs) exist, the total energy of a medium with waves becomes less than the energy of the medium without waves. So, no energy is needed to excite NEWs; instead, some energy can be extracted from the system by the excitation of such waves. The more energy is extracted, the more intensely NEWs are excited in the system.

The concept of waves possessing opposite sign energies in shear flows enables researchers to interpret various types of instabilities in hydrodynamics [3,4]. In particular, the well-known Kelvin–Helmholtz instability can be interpreted in terms of energy exchange between two modes — one of positive energy and another of negative energy [5,4]. It is clear that the NEWs may exist only in non-equilibrium media, in particular, in shear flows with a large stock of kinetic energy. In addition to hydrodynamic flows, there are many other examples of such media supporting NEWs; for example, beams of charged particles in a plasma, two-level atomic systems, etc.

The concept of negative energy waves was proposed by L. Chua as applied to waves in electron beams back in 1951 (see, e.g. [6]) and has since been widely used in hydrodynamics, plasma physics and electronics; numerous examples can be found in [7–10,5,11,4]. Nevertheless, this concept still causes certain difficulties to some researchers, therefore it seems reasonable to consider one more example of practical interest which demonstrates the possibility of existence of NEWs and represents a certain academic interest. We consider below surface gravity-capillary waves on a shear flow with the velocity profile linearly depending on the depth (the Couette-type shear flow with a constant vorticity).

There is another interesting aspect of the problem of wave propagation in shear flows. It is common knowledge that average potential and kinetic energies are equal in both surface and internal waves of small amplitude in a motionless fluid [12] by virtue of the virial theorem [13] for arbitrary mechanical systems. An average value of the Lagrangian function [14] that coincides with the difference between the kinetic and potential energies is equal to zero in this case. However, average kinetic and potential energies do not necessarily coincide in non-equilibrium media such as, for example, stratified fluids with shear flows. Actually, the kinetic energy density in a shear flow depends on the strength of the background

* Corresponding author.
E-mail address: Yury.Stepanyants@usq.edu.au (Y. Stepanyants).

[1] The authors adhere the principle of alphabetical order of names.

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flow (this will be specified below), whereas the potential energy is independent of a shear flow and determined solely by the displacement of the free surface (and isopycnic surfaces in a stratified fluid). In this case, the Lagrangian function is no longer determined by the difference between the kinetic and potential energies although the average value of the Lagrangian is still equal to zero. The latter may occur because the average Lagrangian of linear perturbations is proportional to the function $D(\omega; k)$, where $D(\omega; k)$ is proportional to the dispersion relation for small-amplitude perturbations [14,5]. Such systems are well known in mechanics and are referred to as non-natural systems [15].

In Ref. [16] was investigated the relation between the kinetic and potential energies for internal waves of infinitesimal amplitude at the interface between two infinitely thick layers of fluids with different densities moving with constant velocity with respect to each other (see also Sect. 1.9 in the book [4]). The problem was analysed both with and without surface tension effect between the layers. The explicit expressions for kinetic and potential energies have been derived and it has been shown that under a certain condition the wave energy may become negative.

The wave energy for surface gravity-capillary waves on a uniform flow has been also calculated by Dysthe [17] for a fluid of a finite depth. The analysis of results obtained can reveal the existence of NEWs in such system, but this issue has not been considered in Dysthe's Lecture Notes.

In the recent publication by Ellingsen and Brevik [18] the wave energy of surface gravity waves on a shear flow with linear velocity profile has been calculated for a fluid of a finite depth. But the authors obtained an incorrect result (the source of the error will be elucidated below) and did not discuss the possibility of existence of NEWs.

Below we calculate the total energy of small-amplitude surface gravity-capillary waves on a shear flow with a linear velocity profile in a fluid of a finite depth. Then we consider the relationship between the kinetic and potential energies for such waves and show that they are not equal in general. With the surface tension effect taken into account we derive the condition when NEWs appear in the system. This problem represents not only an academic interest, but may be important for practical applications. The results obtained can be further developed in application to the study of wave blocking phenomena when a shear flow gradually varies in the horizontal direction.

2. Problem statement and the dispersion relation

Let us consider one-dimensional wave propagation on a surface of moving water of a finite depth h. We assume that the velocity profile of the water flow $U(z)$ is a linear function of the depth $U(z) = U_0 + \alpha z$. Here U_0 is the water speed at the free surface, and α characterises the vorticity of the background flow. When the fluid velocity vanishes at the bottom, then $\alpha = U_0/h$, but in general α may be an independent parameter; in particular, putting $\alpha = 0$ we obtain the uniform current without vorticity. The shear flow vorticity can be controlled in the finite depth fluid with the help of a movable bottom, for example, by using a rubber conveyor. It is assumed that the axis z is directed upward with a zero at the unperturbed water surface. For certainty we suppose that $U_0 > 0$, i.e. the background flow at the water surface is co-directed with the axis x; in other words, the velocity vector is $\mathbf{U}(z) = U_0(z)\hat{\mathbf{i}}$. The sketch of the flow considered here is shown in Fig. 1.

The dispersion relation for surface waves in water with a current linearly varying with depth has been derived both for pure gravitational waves [19–22] and for capillary–gravity waves [23,24,18]. We will re-derive it below in the form convenient for our analysis. The main aim of this paper is to derive in the linear approximation the wave energy of surface gravity-capillary waves

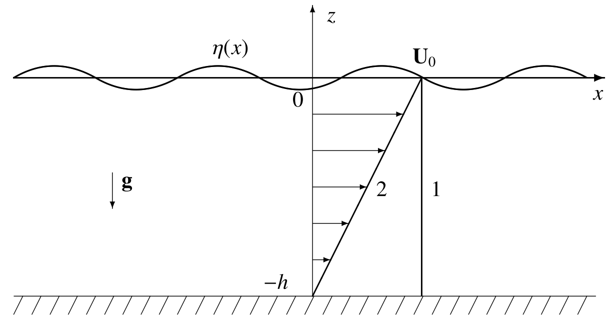

Fig. 1. Sketch of the fluid flow in the reference coordinate frame associated with the immovable bottom. Line 1 depicts the velocity with the uniform profile and line 2 the velocity with the linear profile.

propagating on a shear flow and demonstrate some interesting nontrivial features related to it. In particular, we show that under certain conditions the wave energy may become negative which indicates that the flow may become unstable. The existence of negative energy waves in uniformly moving media is well-known (see the references listed above), but to the best of our knowledge, the influence of the basic flow vorticity on the wave energy has not been studied thus far. In our study we obtain the general results which account for the arbitrary flow vorticity and naturally reduce to the case of a uniformly moving fluid. We calculate explicitly the ratio of kinetic to potential energies for a linear wave and analyse in detail its dependence on the flow intensity, vorticity and surface tension. We also show that the virial theorem of the energy equipartition between these two energy components is not valid for waves on a linear shear flow.

2.1. Basic equations and derivation of the dispersion relation

For our purposes it is convenient to present the Euler equation in the Helmholtz form [25]:

$$\frac{\partial(\mathrm{curl}\,\mathbf{v})}{\partial t} = (\mathrm{curl}\,\mathbf{v}\cdot\nabla)\,\mathbf{v} - (\mathbf{v}\cdot\nabla)\,\mathrm{curl}\,\mathbf{v}. \tag{1}$$

In a two-dimensional flow when all variables depend on two spatial coordinates x and z, the curl of any vector is perpendicular to the x, z-plane, whereas operator nabla has components in that plane. Therefore the first term on the right-hand side of Eq. (1) is zero, and the equation reads

$$\frac{\partial(\mathrm{curl}\,\tilde{\mathbf{v}})}{\partial t} = -(\mathbf{v}\cdot\nabla)\,(\mathrm{curl}\,\mathbf{U} + \mathrm{curl}\,\tilde{\mathbf{v}}), \tag{2}$$

where $\tilde{\mathbf{v}}$ is the perturbation of the velocity field, and $\mathbf{U}(z)$ is the background flow with the linear profile shown in Fig. 1; its curl is constant and directed perpendicular to the x, z-plane, $\mathrm{curl}\,\mathbf{U} = \alpha\,\hat{\mathbf{j}}$ (the flow with constant vorticity). In this case the time derivative of the basic vorticity vanishes, and the velocity perturbation remains potential if it was potential initially. This allows us to consider the potential velocity perturbation $\tilde{\mathbf{v}} \equiv (u, v) = \nabla\varphi$ superimposed on the background flow of constant vorticity. Note that this is the only case of vortical flow which is compatible with potential perturbations [26]. Then we have the following equation for the velocity potential in the entire fluid domain:

$$\Delta\varphi = 0. \tag{3}$$

The boundary conditions are conventional: there is no water flow through the rigid bottom, therefore

$$\frac{\partial\varphi}{\partial z} = 0 \quad \text{at } z = -h. \tag{4}$$