European Journal of Mechanics B/Fluids 53 (2015) 160–170

Contents lists available at ScienceDirect

European Journal of Mechanics B/Fluids

journal homepage: www.elsevier.com/locate/ejmflu

Initial stages of the interaction between uniform and pointwise vortices in an inviscid fluid

Giorgio Riccardi *

Department of Industrial and Information Engineering, Second University of Naples, via Roma 29, 81031 Aversa (CE), Italy
CNR-INSEAN, National Research Council of Italy, Maritime Research Center, via di Vallerano 139, 00128 Rome, Italy

HIGHLIGHTS

- The motion of a uniform vortex in presence of a point vortex is investigated.
- The vortex shape is considered by means of the Schwarz function of its boundary.
- The equation of the dynamics of the Schwarz function is employed.
- Successive approximations are used to analytically approximate the solution.
- Results agree with numerical simulations at small times.

ARTICLE INFO

Article history:
Received 16 February 2015
Received in revised form
29 March 2015
Accepted 28 April 2015
Available online 19 May 2015

Keywords:
Two-dimensional vortex dynamics
Vortex interaction
Schwarz function
Singular integral equation

ABSTRACT

The motion of a uniform vortex in presence of a pointwise one in an isochoric, inviscid fluid is analytically investigated. The uniform vortex is initially circular and the point vortex lies inside or outside this circle. At successive times, the shape of the uniform vortex is accounted for by means of the Lagrangian form of the Schwarz function of its boundary. A novel mathematical approach is adopted, based on the time evolution equation of this function. It leads to a non-linear singular integral system, the analytical solution of which is addressed by means of successive approximations. The 0th order one neglects the non-linear terms, while the kth ($k \geq 1$) approximation accounts these terms as forcing ones, once they are evaluated in correspondence to the $(k-1)$th approximation. However, due to the increasing algebraic difficulties in handling these approximations, the present analysis is limited to the 1st order one. In several sample cases its description of the motion is compared to the fully non-linear numerical simulation and a satisfactory agreement is found, at least for small times.

© 2015 Elsevier Masson SAS. All rights reserved.

1. Introduction

The present paper deals with the two-dimensional dynamics of a uniform vortex in presence of an internal or an external point vortex. The fluid is assumed as inviscid and isochoric. The subject is quite old: it has been deeply investigated in Literature by means of numerical simulations as well as of several simplified models, aimed to explain the principal features of this vortex interaction. In particular, when the point vortex lies inside the uniform one the investigations have been focused on the excitation of Kelvin waves travelling on the vortex boundary by the point vortex and on the interaction between these waves and the point vortex itself (*e.g.*, the wave breaking and the consequent filamentation). Instead, when the point vortex is external to the uniform one a lot of analyses concern the merging mechanism. An attempt to give some basic physical ideas about these investigations may be the following one.

Among many others, a model for studying the motion of point vortices inside a uniform one (at vorticity level ω) is built in [1], with the aim to investigate the breaking of the Kelvin waves, that triggers the filamentation of the vortex boundary. The basic hypotheses of this model are the following ones. The uniform vortex is circular at the initial time (with centre on the origin) and its circulation is much larger than the ones of the point vortices (or $\lambda :=$ mean |point vortex circulations|/uniform vortex circulation $\ll 1$). Moreover, the boundary of the uniform vortex allows the polar description $r(\theta; t) = 1 + \varepsilon(\theta; t)$, r and θ being the distance from the origin and the azimuthal angle (lengths and times are non-dimensionalized by the initial radius of the vortex and by its

* Correspondence to: Department of Industrial and Information Engineering, Second University of Naples, via Roma 29, 81031 Aversa (CE), Italy.
E-mail address: giorgio.riccardi@unina2.it.

http://dx.doi.org/10.1016/j.euromechflu.2015.04.009
0997-7546/© 2015 Elsevier Masson SAS. All rights reserved.

eddy turn-over time $4\pi/\omega$). The perturbation ε is assumed of order λ and represented in Fourier series. In this way, approximate (to order λ^2) differential equations for the time evolution of each Fourier mode and point vortex position are written, in a reference system rotating with the unperturbed circular vortex. This complicated Cauchy problem is then simplified by means of the following key-observations. The free dynamics of the uniform vortex boundary occurs at the timescale t: the Kelvin wave corresponding to the kth Fourier mode propagates (in clockwise direction) with a phase velocity of the order of the unity (the linear analysis gives $-2\pi(|k|-1)/|k|$). On the other hand, each point vortex moves with the sum of the velocities induced by the waves propagating on the boundary and by the remaining vortices, these latter being of order λ. As a consequence, the point vortex motion basically occurs in the slow timescale λt, being perturbed by the action of the Kelvin waves propagating at the fast timescale t. These observations suggest to perform an analysis with two timescales: a fast (t) and a slow (λt) one, taking advantage of the so called "resonance condition" to evaluate the slow-varying terms. It is found that a "fast" filamentation occurs within one rotation period, if one of the point vortices approaches the boundary at a distance of the order $\lambda^{0.566}$, while if all the vortices are far from the boundary the vortex can experience a "slow" filamentation, at times of order $-(\log\lambda)/\lambda$.

The results of the previous theoretical analysis have been experimentally confirmed by visualizing the motion of a strongly magnetized electron column confined in a Malmberg–Penning trap and using the electromagnetic analogy [2]. As discussed before, a point vortex moving inside a uniform one interacts with the Kelvin waves travelling on the boundary. Eventually, these interactions lead to the wave breaking and to the filamentation of that curve. The experiments performed in the above paper enable to verify the scaling law of the distance of a vortex from the boundary needed for the "fast" (within a period of rotation of the uniform vortex) filamentation, as well as to measure the breaking time of a "slow" one. Experimental results agree in both cases with the scaling laws: *distance* $\sim \lambda^{0.566}$ and *breakingtime* $\sim -(\log\lambda)/\lambda$ predicted in [1]. Moreover, the entrapment of irrotational flow islands inside the uniform vortex, due to the reattachment of the filament to the vortex core, is also visualized.

As stated above, analyses of the motion with the point vortex outside the uniform one concern essentially the study of the merging mechanism. In the important paper [3], the merging has been explained in terms of resonances between the point vortex motion and the Kelvin waves on the vortex boundary. The analysis is developed in the basic hypotheses that the circulation of the point vortex is much smaller than the one of the uniform vortex, *i.e.* their ratio γ is assumed $\ll 1$. For any Kelvin wave of azimuthal number m (note that m must be ≥ 2), a critical radius r_m is defined, as the one at which the most efficient excitation of the surface wave takes place. This condition is reached when the rotation period of the point vortex placed at distance r from the centre of the uniform one, *i.e.* r^2 (lengths and times are non-dimensionalized as before), is just m times the wave period, *i.e.* $1/(m-1)$. It follows $r_m = [m/(m-1)]^{1/2}$. What happens near a critical radius? To answer to this question, the point vortex is placed at a distance $r_0 \simeq r_m$ from the centre of the uniform one and the Hamiltonian of the system is approximated by accounting for the point vortex and the mth Kelvin wave, only. It becomes very simple once its is rewritten in terms of the rescaled radius $\tilde{r} := (r_0 - r_m)/\delta r_m$, the constant δr_m being proportional to $\gamma^{1/3}$. The phase portrait of the approximate Hamiltonian shows that two kinds of trajectories are possible: rotation around the uniform vortex and nutations. These latter imply finite amplitude oscillations (of order 1 in $\tilde{r}$) in the radial direction and the motion of the point vortex occurs into a neighbourhood of the critical radius r_m, that is called "critical layer".

An interesting merging scenario follows from this analysis. When the relative intensity γ is very low, δr_m is much smaller than $r_m - r_{m+1}$, the critical layers are well separated and the motion of the vortex is confined inside one of them. On the contrary, if the point vortex is intense enough, δr_m becomes of the same order than $r_m - r_{m+1}$, so that a nutation in the mth critical layer moves the vortex inside the neighbouring $m+1$ layer, where it excites the $(m+1)$th Kelvin wave and so on. In this way, the point vortex spirals in the uniform one and Kelvin waves of higher and higher wavenumbers are excited, mimicking the filamentation of the boundary.

This merging scenario has been numerically confirmed in [4], still by means of the electromagnetic analogy. The paper explores also the merging interactions in which the intensity of the point vortex is comparable with the one of the uniform vortex. The merging is rather rapid and its mechanism is completely different from the previous ones. Indeed, the point vortex wraps the uniform one around itself, by trapping irrotational fluid and forming a complicated vortex spiral.

In the last years, a large research activity has been devoted to build and analyse stationary solutions of the Euler equation, in terms of uniform and pointwise vortices. Due to the fact that the mathematical device used in these analyses is rather similar to the one employed in the present paper, they are now briefly described. By placing symmetrical configurations of point vortices into a uniform one, multipolar equilibria have been built in [5]. In this paper, the Schwarz function of the boundary of the uniform vortex ($\boldsymbol{\Phi}$, see Section 2 for details) is used, by exploiting its relation with the streamfunction in a corotating frame of reference:

$$\psi_r(\boldsymbol{x}) \propto \boldsymbol{x}\overline{\boldsymbol{x}} - \int^{\boldsymbol{x}} d\boldsymbol{y}\boldsymbol{\Phi}(\boldsymbol{y}) - \int^{\overline{\boldsymbol{x}}} d\boldsymbol{y}\overline{\boldsymbol{\Phi}}(\boldsymbol{y}). \tag{1}$$

The stability properties of these equilibria have been discussed in [6], by using linear analyses as well as simulations of the flow, performed by means of contour dynamics [7]. Quadrupoles, pentapoles and higher have been found to be stable equilibria, while the tripoles results to be linearly unstable. Other interesting equilibria involving a doubly connected uniform vortex and an internal set of pointwise ones are found in [8], mimicking the overlapping of shielded Rankine vortices. Here an irrotational region remains also trapped at the centre of the vortex. In [9] a family of stationary solutions given by a uniform vortex surrounded by a certain number of corotating point vortices, placed on the vertices of a regular polygon is built and analysed. In [10] the streamfunction (1) is used to generalize to finite-area vortices the ideas of Aref and Vainchtein [11], who search asymmetric equilibria of point vortices by inserting new vortices on points of rest in a corotating reference frame. Growing uniform vortices are inserted in a corotating vortex pair, until the Rankine vortex is reached. Other equilibria involving uniform and point vortices are found in [12], still starting from the streamfunction (1). A central uniform vortex is surrounded by an alternate distribution of pointwise and uniform vortices. Vortex shapes with cusps are found and contour dynamics simulations show the formation of filaments in configurations having large satellite vortices with cusps, as well as in the perturbed equilibria obtained by displacing the point vortices.

The present paper tries to open a new way towards a theoretical understanding of such kind of flows, by describing the motion of the uniform vortex in terms of the Schwarz function of its boundary. As well known, this function contains all the information about the shape of such a closed, simple curve. It can be easily calculated from the analytical form of the vortex boundary at the initial time, but (to the best of the author knowledge) there is no way to find it at any successive time, in particular by analysing the results of a numerical simulation. The consequent loss of information is one of the main responsible for the lack of mathematical and physical understanding of the flow. In order to partially

Download English Version:

https://daneshyari.com/en/article/7051315

Download Persian Version:

https://daneshyari.com/article/7051315

Daneshyari.com