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Wave motion in a heavy compressible fluid: Revisited

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HIGHLIGHTS

- The nonlinear interaction of gravity and acoustic modes is studied.
- A solution to the Longuet-Higgins resonance case and its evolution is obtained.
- The results enrich our understanding of microseisms generation.

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ABSTRACT

The effect of the compressibility of the ocean and its role in generating progressive compression-type waves (acoustic-gravity waves) are revisited. Originally, Longuet-Higgins (1950) obtained a solution for the nonlinear interaction of two opposite waves in a compressible fluid, which results in the generation of compression waves. In this paper we extend this solution to include the general interaction of waves of profoundly different wavelengths. We also fully solve the special triad resonance case obtained by Longuet-Higgins (1950).

1. Introduction

The small compressibility of water is negligible in the overwhelming majority of ocean surface-waves studies. In an incompressible ocean, with constant depth h, the solution of the field equation (i.e. the *Laplace* equation), results in a single progressive surface wave, for each prescribed frequency ω and corresponding wave number k. However, accounting for the small compressibility of the water gives rise to several progressive waves with wave numbers k_n, $n = 0, 1, \dots, N$. The wave number k_0 corresponds to the progressive wave to be addressed as *gravity* wave, and it is almost identical to k; the wave numbers $k_1 \dots k_N$ are much smaller than k_0, correspond to compression-type waves also known as *acoustic-gravity* waves, where N is the nearest integer smaller than $(\omega h/\pi c + 1/2)$, and c is the speed of sound in water, assumed constant. For the sake of brevity, and apart from the gravity wave, we consider the first acoustic-gravity mode only. Compared to other acoustic-gravity modes, the first mode has normally the largest amplitude [1], containing most energy [2]. Moreover, it is the only mode that outlasts below a critical depth, turning into a Scholte wave and then into a Rayleigh wave at the shoreline [2].

The aim of the current work is to provide a solution for the nonlinear interaction of waves of different periods and their effect on both gravity and acoustic-gravity modes, in a compressible fluid of finite depth. In addition, a solution to the Longuet-Higgins resonance case (see [3],[1] Section 4, Equation (176)) and its evolution is provided. The obtained results enrich our understanding on the generation and evolution of compression waves—the origin of microseisms.

The first theoretical works on the generation of acoustic-gravity modes by fluid flows were described by *LH* and Lighthill [4], and later extended by Hasselmann [5] and Hasselmann [6] who studied the problem from a statistical point of view. It has been commonly accepted that the mechanism for the generation of low frequency acoustic-gravity waves in the ocean is the nonlinear interaction of two gravity waves that travel in nearly opposite directions [7,8]. The generation of acoustic-gravity waves by nonlinear interaction of gravity waves has been studied extensively by Kibblewhite [9–12] and Webb [13–16]. More recently, this subject has also been studied by Ardhuin and Herbers [17], Kadri and Stiassnie [18], Ardhuin et al. [19] and Duennebier et al. [20].

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In this paper we extend the work reported in Section 4 of *LH* to include interaction of waves with wavelengths not necessarily equal. For the sake of brevity, and in order to allow for a straightforward comparison with *LH*, we use identical notation and analysis where applicable. The obtained solution is valid for waves of profoundly different wavelengths. The basics and formulation of the problem are given in Section 2. The first and second order solutions are presented in Sections 3 and 4, respectively. The solution of the *LH* resonance case is given in Section 5. Finally, the concluding remarks are given in Section 6.

2. General equations

Consider the two dimensional problem of surface waves interacting on an ideal compressible fluid of a constant depth. The flow is assumed irrotational. A Cartesian system (x, z) with the origin in the undisturbed free surface and the z-axis vertically upwards is considered. The governing equation is the two dimensional compressible wave equation (e.g., [21]; *LH*)

$$\frac{\partial^2\phi}{\partial t^2} - c^2\nabla^2\phi + g\frac{\partial\phi}{\partial z} = -\frac{\partial}{\partial t}\left(\mathbf{u}^2\right) \quad \text{on } -h < z < \eta, \tag{1}$$

where $\phi(x, z, t)$ is the flow velocity potential, and the velocity of the fluid is defined by the gradient of its potential ($\mathbf{u} = \text{grad}\phi$); and $z = -h$ is the equation of the rigid bottom. The bottom boundary condition is

$$\frac{\partial\phi}{\partial z} = 0, \quad \text{on } z = -h. \tag{2}$$

Assuming a constant pressure at the free surface $z = \eta$, and using Taylor's expansion around $z = 0$ gives

$$\left(\frac{\partial\phi}{\partial t} - \frac{1}{2}\mathbf{u}^2\right) + \eta\left(\frac{\partial^2\phi}{\partial t\partial z} - \mathbf{u}\cdot\frac{\partial\mathbf{u}}{\partial z} + g\right) + \cdots = 0, \quad \text{on } z = 0, \tag{3}$$

and the kinematic boundary condition reduces to

$$\left(\nabla^2\phi\right) + \eta\left(\frac{\partial}{\partial z}\nabla^2\phi\right) + \cdots = 0, \quad \text{on } z = 0, \tag{4}$$

where $(\ldots)$ denotes higher order terms.

Following *LH*, using the method of successive approximations we define

$$\phi = \epsilon\phi_1 + \epsilon^2\phi_2 + \cdots, \qquad \mathbf{u} = \epsilon\mathbf{u}_1 + \epsilon^2\mathbf{u}_2 + \cdots, \qquad \eta = \epsilon\eta_1 + \epsilon^2\eta_2 + \cdots, \quad \text{etc.}, \tag{5}$$

where ϵ is a small ordering parameter. Substituting definitions (5) into Eqs. (1), (2), and (4), and taking terms of order ϵ yields

$$\frac{\partial^2\phi_1}{\partial t^2} - c^2\nabla^2\phi_1 + g\frac{\partial\phi_1}{\partial z} = 0, \quad \text{on } -h < z < \eta, \tag{6}$$

$$\frac{\partial\phi_1}{\partial z} = 0, \quad \text{on } z = -h, \tag{7}$$

$$\nabla^2\phi_1 = 0, \quad \text{on } z = 0, \tag{8}$$

where

$$\mathbf{u}_1 = \text{grad}\phi_1, \tag{9}$$

$$g\eta_1 = -\frac{\partial\phi_1}{\partial t}, \quad \text{on } z = 0. \tag{10}$$

Similarly, for the second order approximation we have

$$\frac{\partial^2\phi_2}{\partial t^2} - c^2\nabla^2\phi_2 + g\frac{\partial\phi_2}{\partial z} = -\frac{\partial}{\partial t}\left(\mathbf{u}_1^2\right), \quad \text{on } -h < z < \eta, \tag{11}$$

$$\frac{\partial\phi_2}{\partial z} = 0, \quad \text{on } z = -h, \tag{12}$$

$$\nabla^2\phi_2 = -\eta_1\left[\frac{\partial}{\partial z}\left(\nabla^2\phi_1\right)\right], \quad \text{on } z = 0, \tag{13}$$

where

$$\mathbf{u}_2 = \text{grad}\phi_2, \tag{14}$$

$$g\eta_2 = -\left[\frac{\partial\phi_2}{\partial t} - \frac{1}{2}\mathbf{u}_1^2\right] - \eta_1\left(\frac{\partial^2\phi_1}{\partial z\partial t}\right), \quad \text{on } z = 0. \tag{15}$$

3. Solution of the first order (linear) approximation

Assume that ϕ_1 is a simple progressive wave given in a complex exponential form

$$\phi_1 = F(z)e^{-i(kx-\omega t)}, \tag{16}$$

where k and ω are real, and $F(z)$ is complex and a function of the vertical axis z only. Writing

$$F(z) = f(z)e^{\gamma z}, \quad \gamma = \frac{g}{2c^2}, \tag{17}$$

and substituting in Eq. (16) and then in (6) yields

$$\frac{\mathrm{d}^2 f(z)}{\mathrm{d}z^2} - \kappa^2 f(z) = 0, \tag{18}$$

where $\kappa^2 = k^2 - \omega^2/c^2 + \gamma^2$. Assuming $\kappa \neq 0$ we have

$$f(z) = Ae^{\kappa z} + Be^{-\kappa z}, \tag{19}$$

where A and B are constants, and thus

$$\phi_1 = Ae^{(\gamma+\kappa)z} + Be^{(\gamma-\kappa)z}. \tag{20}$$

From conditions (7) and (8) we obtain the following two equations

$$\begin{aligned}&(\gamma+\kappa)e^{-(\gamma+\kappa)h}A + (\gamma-\kappa)e^{-(\gamma-\kappa)h}B = 0,\\ &\left[(\gamma+\kappa)^2 - k^2\right]A + \left[(\gamma-\kappa)^2 - k^2\right]B = 0.\end{aligned} \tag{21}$$

The determinant of this homogeneous system is

$$\Delta(\omega, k) = (\gamma+\kappa)\left[(\gamma-\kappa)^2 - k^2\right]e^{-(\gamma+\kappa)h} - (\gamma-\kappa)\left[(\gamma+\kappa)^2 - k^2\right]e^{-(\gamma-\kappa)h}. \tag{22}$$

Note that $\Delta(\omega, k) = 0$, which is required for a non-trivial solution of (22), yields the dispersion relation[2]

$$\omega^2 = g\frac{\kappa^2 - \gamma^2}{\kappa\coth\kappa h - \gamma}. \tag{23}$$

An approximate explicit dispersion relation that was added in proof is given in Appendix A.

The solution of the first order approximation is finally given by

$$\phi_1 = -\frac{H}{2}\frac{g}{\omega}\frac{\kappa\cosh\left[\kappa\,(h+z)\right] - \gamma\sinh\left[\kappa\,(h+z)\right]}{\kappa\cosh(\kappa h) - \gamma\sinh(\kappa h)} \times e^{\gamma z}\sin(kx - \omega t), \tag{24}$$

[2] Neglecting gravity effects the relation (23) reduces to $\omega^2 = g\kappa\tanh\kappa h$, and in the incompressible limit, $\kappa = k$, it reduces to the well-known dispersion relation $\omega^2 = gk\tanh kh$, where $k = k_0$ is the gravity mode.