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On periodic solutions of interfacial waves of finite amplitude

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HIGHLIGHTS

- Periodic finite amplitude waves between two fluid layers are shown to exist.
- The weakly nonlinear models are inaccurate when considering the periodic case.
- Finite amplitude periodic internal waves have a slower decay rate.
- Finite amplitude periodic internal waves have a higher threshold velocity.
- Finite amplitude periodic internal waves can be approximated as a sum of soliton shapes.

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ABSTRACT

Using the two-layer fluid model for long interfacial waves of finite amplitude, we find solutions for periodic traveling waves and investigate their properties. In addition, it is shown that these periodic traveling waves can be represented as an infinite sum of spatially repeated soliton shapes, although it is concluded that this is merely a very accurate approximation and not a mathematical property of the model (as is often characteristic for cases of weak nonlinearity). By reducing the aforementioned model to its small amplitude counterpart (i.e. the Benjamin–Ono equation), an explanation is provided as to the fact that even for small wave amplitudes, there remains an apparent discrepancy between the models which increases as the period length shortens.

1. Introduction

Traveling nonlinear (solitary and periodic) waves are waves which propagate on the interface bounding two immiscible fluids without changing their shape. This phenomenon is possible due to a delicate balance between dispersion and nonlinear effects, such that for a specific wave shape, all of its components will advance at the same rate. One important class in this category is that of traveling gravity waves, for which several models exist. There is a vast literature on theoretical, numerical and experimental investigations of interfacial and internal solitary waves and here we list only few [1–19]; however, it is important to note that most of these studies primarily deal with *solitary* solutions and do not discuss the possibility of *periodic* waves. This particular point is further elaborated on in this paper. As far as periodic nonlinear waves are concerned, it is worth noting that the subject of *generalized* solitary solutions for internal waves (including the case of continuous stratification) has also been elaborated, even in the large amplitude regime (see for example [20–23]). Typical generalized solitary waves are characterized by a single pulse plus a periodic ripple wave structure connecting to infinity. In this work however we choose to concentrate on the periodicity of long (baroclinic) interfacial finite-amplitude wave solutions in a planar two-layer fluid system.

Weakly nonlinear models for long waves (i.e., models assuming small wave amplitude and large typical wave length in comparison with the fluid layer depth) are amongst the most investigated due to the fact that they are integrable and may provide analytical insight into many physical phenomena, such as surface and internal waves in the ocean and atmosphere. The three main 'weak' models often used are the Korteweg–de Vries (KdV) model – which is applicable to surface waves traveling over a single shallow layer – and the Benjamin–Ono (BO) and Intermediate Long Wave (ILW) models which deal with internal waves traveling at the interface between two layers: one shallow and the other deep (see for example [6]). All three models are known to support both solitary and periodic traveling waves, with the former comprising a limiting

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case of the latter. However, the assumption of weak nonlinearity has been shown to be problematic when describing naturally arising phenomena, particularly in the case of the two- layer model, which has failed to accurately describe experimental results outside a narrow parameter space [12].

This shortcoming makes it desirable to use models which omit the assumption of weak nonlinearity, giving rise to new and more precise descriptions of long traveling gravity waves [8]. This precision is of particular importance in the case of internal waves where the validity of the weakly nonlinear models is limited to a small amplitude range. Indeed, part of the motivation for searching for interfacial periodic solutions of finite amplitude is the fact that internal periodic weakly nonlinear waves practically do not exist for any arbitrary propagation speed but instead have a distinct threshold velocity, under which no solutions are to be found. For relatively short wave lengths, this threshold velocity can be well beyond the limits of validity for the weakly nonlinear models, thus making it crucial to use finite amplitude models in order to correctly describe the periodic waves. This observation has not been reported nor discussed before in the context of nonlinear interfacial or internal waves.

The objectives of the current work are thus to investigate the general behavioral properties of periodic interfacial waves of finite amplitude – as was previously done for the case of internal solitary waves – and by doing so explain the differences that arise when compared against their weakly nonlinear counterparts. A unique and rather unexpected feature of weakly nonlinear integrable models is that given the solitary wave solution, a periodic wave solution can be constructed using an infinite row of equally spaced solitary wave shapes. Nevertheless, the speed of the periodic wave train is different from the corresponding velocity of each soliton because of nonlinear interaction. This feature was first proven by Toda [24] to hold for nonlinear lattices and later on by [25,26] for the KdV and modified KdV models. Further extensions for both BO and ILW models are also known [5,6,27]. A similar superposition principle has been also established for wave solution of the Camassa–Holm equation which are characterized by discontinued first derivatives [28,29]. It is interesting to note that such a superposition dictum can be shown to exist also for partially integrable [30] or even nonintegrable nonlinear evolution equations which arise in other branches of mathematics and physics [31,32]. The common feature to all these 'weak' models is that a solitary wave solution can be analytically found and thus a particular superposition construct can be rigorously proven. Yet, an open question which still remains and also dealt in the sequel is determining whether such a compelling feature is also true for finite *(non-weak)* amplitude interfacial waves as in the present case.

The structure of this note is as follows: in Section 2 we present a short derivation of the corresponding finite amplitude model following the lines of [8], including a comparison to the BO and ILW 'weak' models. In particular, we address the issue of the existence of a threshold velocity for the finite-amplitude periodic wave solutions under limiting cases. A straightforward numerical scheme for computing periodic waves is further developed and presented in Section 3. Some numerical simulations of periodic wave solutions for various flow and geometric parameters are finally discussed in Section 4, including a detailed analysis of the dependence of the critical (threshold) minimal velocity in terms of the physical parameters. We conclude with a short summary in Section 5 and a discussion of the validity of the repeated soliton '*summation*' property for the present 'non-weak' case.

2. The two-layer finite-amplitude model and preliminary analysis

Beginning with Euler's equations and following Choi and Camassa [8], one can show that the model equations for two-dimensional long internal waves of finite amplitude at the interface between two immiscible fluid layers bounded with rigid lids are given by

$$\begin{aligned} &\eta_t + (\eta\bar{u})_x = 0 \\ &\bar{u}_t + \bar{u}\bar{u}_x - g^*\eta_x = \rho_r H\left[(\eta\bar{u})_{xt}\right] \end{aligned} \tag{2.1}$$

where $\eta = h - h_1$ is a measure of the wave height h with respect to the shallow upper prescribed layer depth h_1, $\bar{u}$ is the averaged horizontal velocity over the depth of the shallow layer and $g^* = g(\rho_r - 1)$ is a modified acceleration of gravity dependent upon the density ratio of the two layers $\rho_r \triangleq \frac{\rho_2}{\rho_1} \geq 1$ where ρ_1 and ρ_2 denote the upper and lower layer densities respectively. Here x denotes the horizontal coordinate and t is the time. The operator $H[f]$, which appears on the right-hand side of the second equation in (2.1), is an integral operator of the following form:

$$H[f(x)] = \frac{1}{2d} p.v. \int_{-\infty}^{\infty} \coth\left(\frac{\pi(x-y)}{2d}\right) f(y)\,dy, \tag{2.2}$$

where d denotes the depth of the deep lower layer, and the acronym $p.v.$ implies that the integral should be interpreted according to Cauchy's *principal value*. In the case of an infinitely deep lower layer, (2.2) reduces to the well-known Hilbert transform

$$H[f(x)] = \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy. \tag{2.3}$$

Eq. (2.1) constitutes a bi-directional model for long interfacial waves and can be considered as a particular case of the nonlinear Su–Gardner (SG) [2] or the one dimensional version of the Green–Naghdi (GN) [3] corresponding models. No restrictions have been imposed so far upon the wave amplitude, so that the above formulation remains valid also for waves of finite amplitude. Assuming no a priori knowledge regarding the model equations for weakly nonlinear waves, we adopt a formal perturbation series approach and define

$$\begin{aligned} &\eta = \eta_0 + \delta\eta_1 + \delta^2\eta_2 + \cdots \\ &\bar{u} = u_0 + \delta u_1 + \delta^2 u_2 + \cdots \end{aligned} \tag{2.4}$$

where $u_0 = \text{const}$; $\eta_0 = \text{const}$; $\delta = \frac{\eta_{\max} - \eta_0}{\eta_0} \ll 1$. Additionally, we define new set of stretched coordinates as;

$$\begin{aligned} &\tilde{x} = \delta(x - c_0 t) \\ &\tilde{t} = \delta^2 t \end{aligned} \tag{2.5}$$

where c_0 is the infinitesimal velocity of translation.

Substituting (2.4) into (2.1), while using (2.5) and collecting terms in different powers of δ, leads to the following weakly nonlinear model equation:

$$\eta_t + c_0\eta_x + \left(\frac{3}{2}\sqrt{\frac{g^*}{h_1}}\right)\eta\eta_x + \frac{c_0^2\rho_r}{2}\sqrt{\frac{h_1}{g^*}} H(\eta_{xx}) = 0 \tag{2.6}$$

where $c_0 = u_0 + \sqrt{g^* h_1}$. Eq. (2.6) is precisely the ILW equation, which in the case of an infinitely deep lower layer reduces to the BO equation [6].

Let us now narrow our discussion solely to the particular case of traveling waves, which in turn implies

$$\begin{aligned} &\eta = \eta(x - ct) \\ &\bar{u} = \bar{u}(x - ct) \end{aligned} \tag{2.7}$$

where in general $c \geq c_0$.

After substituting (2.7) into (2.1) and (2.6), integrating once and introducing the boundary conditions at infinity, namely: $|x| \to \infty, \eta \to -h_1$ $(h \to 0)$, $u \to u_0$, we respectively obtain the