European Journal of Mechanics B/Fluids 49 (2015) 217–225

Contents lists available at ScienceDirect

European Journal of Mechanics B/Fluids

journal homepage: www.elsevier.com/locate/ejmflu

Asymptotic solutions of weakly compressible Newtonian Poiseuille flows with pressure-dependent viscosity

Stella Poyiadji [a], Kostas D. Housiadas [b], Katerina Kaouri [a], Georgios C. Georgiou [a,*]

[a] Department of Mathematics and Statistics, University of Cyprus, P.O. Box 20537, 1678 Nicosia, Cyprus

[b] Department of Mathematics, University of the Aegean, Karlovassi, 83200 Samos, Greece

ARTICLE INFO

Article history:
Received 11 January 2014
Received in revised form
23 August 2014
Accepted 22 September 2014

Keywords:
Poiseuille flow
Newtonian fluid
Compressible flow
Pressure-dependent viscosity
Perturbation methods
Asymptotic methods

ABSTRACT

We consider both the axisymmetric and planar steady-state Poiseuille flows of weakly compressible Newtonian fluids, under the assumption that both the density and the shear viscosity vary linearly with pressure. The primary flow variables, i.e. the two non-zero velocity components and the pressure, as well as the mass density and viscosity of the fluid are represented as double asymptotic expansions in which the isothermal compressibility and the viscosity–pressure-dependence coefficient are taken as small parameters. A standard perturbation analysis is performed and asymptotic, analytical solutions for all the variables are obtained up to second order. These results extend the solutions of the weakly compressible flow with constant viscosity and those of the incompressible flow with pressure-dependent viscosity. The combined effects of compressibility and the pressure dependence of the viscosity are analyzed and discussed.

1. Introduction

In most isothermal flows of Newtonian liquids, the density and the viscosity are commonly assumed to be constant. Such an assumption, however, is valid only at low processing pressures and may introduce significant error when modeling flows involving high pressures or a large pressure range, such as polymer processing, crude oil and fuel oil pumping, fluid film lubrication, microfluidics, and in certain geophysical flows [1–4].

Waxy crude oil transport [5], polymer extrusion [6,7], and polymer injection molding [8] are important cases of liquid flows in long tubes where compressibility effects cannot be neglected. An exponential equation of state, relating the mass density of the fluid, ρ^*, to the total pressure, p^*, is very often used for compressible liquids [5,9]. For weakly compressible liquids, the following linear equation of state is a good approximation to the exponential equation of state:

$$\rho^* = \rho_0^* \left[1 + \varepsilon^*(p^* - p_0^*)\right] \tag{1}$$

where ε^* is the isothermal compressibility, assumed to be constant, and ρ_0^* is the mass density of the fluid at the reference pressure p_0^*. It should be noted that a superscript star throughout the text indicates a dimensional quantity.

Various numerical solutions for weakly compressible Poiseuille flows for Newtonian [10] as well as non-Newtonian fluids, such as the Carreau fluid [6], the Bingham plastic [5], and certain viscoelastic fluids [11] are available in the literature. Venerus and coworker [12,13] derived analytical perturbation solutions in terms of the compressibility for the axisymmetric and the plane isothermal Poiseuille flow of a weakly, compressible Newtonian liquid respectively, using the steamfunction/vorticity formulation and employing Eq. (1). Taliadorou et al. [14] obtained equivalent solutions using a methodology in which the perturbation is performed on the primary flow variables, i.e. on the velocity components and the pressure. Housiadas and collaborators [9,15,16] extended the primary-variable perturbation method to derive solutions of the plane and axisymmetric Poiseuille flows of a weakly compressible viscoelastic Oldroyd-B fluid.

Flows of fluids with pressure-dependent viscosity have received an increasing attention recently. The viscosity of typical liquids begins to increase substantially with pressure when pressures of the order of 1000 atm are reached [17]. In fact, under certain conditions, e.g. in elastohydrodynamics, the dependence of the viscosity on pressure may be several orders of magnitude stronger than that of density [3,17,18]. Málek and Rajagopal [19] reviewed different equations proposed in the literature in order to describe experimental observations on the pressure-dependence of the

* Corresponding author.
E-mail addresses: map4sp1@yahoo.com (S. Poyiadji), housiada@aegean.gr (K.D. Housiadas), k.kaouri.95@cantab.net (K. Kaouri), georgios@ucy.ac.cy (G.C. Georgiou).

http://dx.doi.org/10.1016/j.euromechflu.2014.09.002
0997-7546/

viscosity. The pressure-dependence of the viscosity in Poiseuille and other flows has been analyzed mathematically by various investigators [17,19–21]. Renardy [17] employed the following linear expression for the viscosity, η^*:

$$\eta^* = \eta_0^* \left[1 + \delta^*(p^* - p_0^*)\right] \tag{2}$$

where δ^* is the viscosity–pressure-dependence material constant and η_0^* is the viscosity at the reference pressure p_0^*. Recently, Kalogirou et al. [22] compiled analytical solutions for unidirectional plane, round, and annular Poiseuille flows of a Newtonian liquid assuming that the viscosity obeys Eq. (2).

In the present work, we consider the steady, isothermal Newtonian Poiseuille flows in a straight channel or slit and in a circular tube, for which both the mass density and the viscosity of the fluid depend weakly on pressure, obeying Eqs. (1) and (2), respectively. To our knowledge, studies taking into account both the compressibility and the viscosity–pressure-dependence are very scarce in the literature. Since exact analytical solutions are not possible, the objective is to obtain approximate analytical solutions for these flows by means of perturbation methods.

The rest of the paper is organized as follows. In Section 2, the governing equations and the boundary conditions are presented. In Section 3, the main steps of the perturbation method are discussed. All flow variables are expressed as double asymptotic expansions in terms of the dimensionless isothermal compressibility and the viscosity–pressure coefficient, which serve as small perturbation parameters. Perturbation solutions are then derived up to second order. The resulting analytical solutions are discussed in Section 4. It is shown in particular that at least up to second order the viscosity–pressure-dependence tends to reduce the velocity in the flow direction and to counterbalance compressibility effects on the pressure. Finally, in Section 5, concluding remarks are provided.

2. Problem and formulation

We consider the steady, weakly compressible isothermal flow of a Newtonian fluid with pressure-dependent viscosity, under zero gravity. The continuity and momentum equations can be written as follows:

$$\nabla^* \cdot (\rho^* \mathbf{u}^*) = 0 \tag{3}$$

$$\rho^* \mathbf{u}^* \cdot \nabla^* \mathbf{u}^* = -\nabla^* p^* + \nabla^* \cdot \boldsymbol{\tau}^* \tag{4}$$

where $\mathbf{u}^*$ is the velocity vector and $\boldsymbol{\tau}^*$ is the viscous extra-stress tensor, given by

$$\boldsymbol{\tau}^* = \eta^*(p^*) \left[\nabla^* \mathbf{u}^* + (\nabla^* \mathbf{u}^*)^T - \frac{2}{3}\mathbf{I}(\nabla^* \cdot \mathbf{u}^*)\right]. \tag{5}$$

In Eq. (5), $\nabla^* \mathbf{u}^*$ is the velocity-gradient tensor, the superscript T denotes the transpose, and $\mathbf{I}$ is the unit tensor. Substituting Eq. (5) into Eq. (4) leads to the following generalized Navier–Stokes equation:

$$\begin{aligned}\rho^* \mathbf{u}^* \cdot \nabla^* \mathbf{u}^* &= -\nabla^* p^* + \eta^* \nabla^{*2} \mathbf{u}^* + \frac{\partial \eta^*}{\partial p^*} \\ &\quad \times \left\{\nabla^* p^* \cdot \left[\nabla^* \mathbf{u}^* + (\nabla^* \mathbf{u}^*)^T\right] - \frac{2}{3}(\nabla^* \cdot \mathbf{u}^*)\nabla^* p^*\right\} \\ &\quad + \frac{\eta^*}{3} \nabla^* (\nabla^* \cdot \mathbf{u}^*).\end{aligned} \tag{6}$$

Two flow geometrical configurations are studied; the first is the axisymmetric Poiseuille flow in a circular tube of constant radius R^* and length L^* in cylindrical coordinates (r^*, z^*), and the second is the planar Poiseuille flow in a straight channel (or slit) of width $2H^*$ and length L^* in Cartesian coordinates (x^*, y^*) centered at the midplane. In the following, we present the axisymmetric case in more detail and provide the most important results for the planar case.

2.1. Axisymmetric flow

For the flow in a circular tube, the governing equations are rendered dimensionless scaling r^* by R^*, z^* by L^*, u_z^* by U^*, u_r^* by $U^* R^*/L^*$, and $p^* - p_0^*$ by $8\eta_0^* L^* U^*/R^{*2}$, where U^* is the mean velocity at the tube exit. The mass density and the viscosity are scaled by ρ_0^* and η_0^*, respectively. Thus, the two components of the momentum equation (6), the continuity equation (3), the equation of state (1), and the equation for the shear viscosity (2) become:

$$\begin{aligned}\alpha Re \rho \left(u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z}\right) &= -8 \frac{\partial p}{\partial z} \\ &\quad + \frac{\eta}{3}\left[\frac{3}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_z}{\partial r}\right) + 4\alpha^2 \frac{\partial^2 u_z}{\partial z^2} + \alpha^2 \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right)\right] \\ &\quad + \frac{2\alpha^2}{3}\frac{\partial \eta}{\partial z}\left[2\frac{\partial u_z}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right] + \frac{\partial \eta}{\partial r}\left(\frac{\partial u_z}{\partial r} + \alpha^2 \frac{\partial u_r}{\partial z}\right)\end{aligned} \tag{7}$$

$$\begin{aligned}\alpha^3 Re\, \rho \left(u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z}\right) &= -8\frac{\partial p}{\partial r} + \alpha^2 \eta \left[\frac{4}{3}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right) + \frac{1}{3}\frac{\partial^2 u_z}{\partial r \partial z} + \alpha^2 \frac{\partial^2 u_r}{\partial z^2}\right] \\ &\quad + 2\alpha^2 \frac{\partial \eta}{\partial r}\left[\frac{\partial u_r}{\partial r} - \frac{1}{3r}\frac{\partial}{\partial r}(r u_r) - \frac{1}{3}\frac{\partial u_z}{\partial z}\right] \\ &\quad + \alpha^2 \frac{\partial \eta}{\partial z}\left(\alpha^2 \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)\end{aligned} \tag{8}$$

$$\frac{\partial(\rho r u_r)}{\partial r} + \frac{\partial(\rho r u_z)}{\partial z} = 0 \tag{9}$$

$$\rho = 1 + \varepsilon p \tag{10}$$

$$\eta = 1 + \delta p \tag{11}$$

where the Reynolds number, Re, the aspect ratio of the tube, α, the dimensionless compressibility number, ε, and the viscosity pressure-dependence number, δ, are respectively defined by:

$$\begin{aligned} Re &\equiv \frac{\rho_0^* U^* R^*}{\eta_0^*}, \qquad \alpha \equiv \frac{R^*}{L^*}, \\ \varepsilon &\equiv \frac{8\varepsilon^* \eta_0^* L^* U^*}{R^{*2}}, \qquad \delta \equiv \frac{8\delta^* \eta_0^* L^* U^*}{R^{*2}}.\end{aligned} \tag{12}$$

The system of equations (7)–(11) closes with appropriate boundary conditions. Along the axis of symmetry, symmetry conditions are applied:

$$\frac{\partial u_z}{\partial r}(0, z) = u_r(0, z) = 0, \quad 0 \le z \le 1. \tag{13}$$

Also, no-slip and no-penetration are imposed along the tube wall:

$$u_r(1, z) = u_z(1, z) = 0, \quad 0 \le z \le 1. \tag{14}$$

Moreover, the pressure datum is set at the tube exit,

$$p(1, 1) = 0 \tag{15}$$

and the dimensionless mass flow rate is unity at any distance $z \in [0, 1]$ from the inlet plane:

$$2\int_0^1 \rho u_z r dr = 1. \tag{16}$$

2.2. Planar flow

The governing equations are rendered dimensionless by scaling x^* by L^*, y^* by H^*, u_x^* by U^*, u_y^* by $U^* H^*/L^*$, and $p^* - p_0^*$ by $3\eta_0^* L^* U^*/H^{*2}$, where U^* is the mean velocity (per unit width) at