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An exact solution for equatorial geophysical water waves with an underlying current

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ABSTRACT

In this paper we present an exact solution to the governing equations for equatorial geophysical water waves which admit an underlying current.

1. Introduction

In this paper we present an exact solution to the β-plane governing equations for geophysical water waves which admit an underlying current. Geophysical ocean waves are those which take into account the Coriolis effects on the fluid body which are induced by the earth's rotation, and the β-plane approximation to the full governing equations applies in regions which are within 5° latitude of the equator [1,2]. The wave solution which we construct in this paper corresponds to steady zonal waves, travelling in the longitudinal direction with a constant speed of propagation $c > 0$, and which experience the presence of a constant underlying current of strength c_0.

Currents, such as the equatorial undercurrent (EUC), feature significantly in the geophysical dynamics of the equatorial region [3,1]. For instance, the El Niño phenomenon has recently been ascribed to the interplay between currents in the ocean and atmosphere [4], and the model we present in this paper is the first approach in incorporating the effects of a current into an exact solution for the governing equations. Additionally, the equator has the remarkable property of acting like a natural waveguide [5]. Accordingly, waves tend to be trapped in the equatorial region, and the waves which we present below inherit this feature—the amplitude of the waves decays rapidly in the meridional direction.

The approach we use to construct these waves is in the spirit of Gerstner's solution for the governing equations of two-dimensional gravity water waves, with significant modifications to incorporate geophysical effects along the lines of [6]. In 1802 Gerstner [7] found an explicit solution in Lagrangian variables for the full water wave equations (the form of this solution was later independently discovered by Rankine). Gerstner's wave is truly remarkable in the mathematical sense that it is one of only a handful of explicit solutions to the full governing equation which have been constructed [8]. Gerstner's wave is a periodic travelling wave with a specific vorticity distribution (see [9,8,10] for a modern treatment of Gerstner's wave). Although the prescription of the flow is quite specific and rigid, remarkably this flow has been recently adapted to describe a wide variety of interesting, and physically varied, water waves (cf. [8,11–13], and particularly [6], where an exact solution to the geophysical governing equations was first derived). We note that all fluid particles follow closed trajectories in Gerstner's wave, something which is precluded for regular irrotational waves [14–22] and which must be due to the underlying vorticity distribution.

The introduction of a current-like term into Gerstner's formulation was performed by Mollo-Christensen [23] in the study of billows between two fluids. Here, we expand this formulation to admit the Coriolis effects of the rotating earth—these effects feature significantly for such large scale phenomena as currents. In

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particular, we find that the introduction of a steady underlying current in the geophysical context has interesting implications for the fluid motion, particularly in relation to the dispersion relation.

2. Governing equations

We take the earth to be a perfect sphere of radius $R = 6378$ km, which has a constant rotational speed of $\Omega = 73.10^{-6}$ rad/s. Then $g = 9.8\ \mathrm{ms}^{-2}$ is the standard gravitational acceleration at the earth's surface, and $\beta = 2\Omega/R = 2.28 \cdot 10^{-11}\ \mathrm{m}^{-1}\ \mathrm{s}^{-1}$ is a parameter which will arise in subsequent considerations [1,2]. From the viewpoint of a rotating reference frame with its origin at the earth's surface, so that the $\{x, y, z\}$-coordinate frame is chosen with z as the vertical variable, x as the longitudinal variable (in the direction due east), and y as the latitudinal variable (in the direction due north), the governing equations for geophysical ocean waves are given by Gallagher and Saint-Raymond [2]

$$u_t + uu_x + vu_y + wu_z + 2\Omega w\cos\phi - 2\Omega v\sin\phi = -\frac{1}{\rho}P_x, \tag{1a}$$

$$v_t + uv_x + vv_y + wv_z + 2\Omega u\sin\phi = -\frac{1}{\rho}P_y, \tag{1b}$$

$$w_t + uw_x + vw_y + ww_z - 2\Omega u\cos\phi = -\frac{1}{\rho}P_z - g, \tag{1c}$$

together with the equation for mass conservation

$$\rho_t + u\rho_x + v\rho_y + w\rho_z = 0 \tag{2a}$$

and the equation of incompressibility

$$u_x + v_y + w_z = 0. \tag{2b}$$

Here the variable ϕ represents the latitude, (u, v, w) is the velocity field of the fluid, ρ is the density of the fluid, and P is the pressure of the fluid. The β-plane approximation of the geophysical governing equations applies when we are working in regions which are within 5° latitude of the equator. There, the latitude ϕ is small and hence the approximations $\sin\phi \approx \phi$, $\cos\phi \approx 1$ are valid, resulting in the β-plane governing equations [2]

$$\begin{aligned} &u_t + uu_x + vu_y + wu_z + 2\Omega w - \beta y v = -\frac{1}{\rho}P_x, && \text{(a)}\\ &v_t + uv_x + vv_y + wv_z + \beta y u = -\frac{1}{\rho}P_y, && \text{(b)}\\ &w_t + uw_x + vw_y + ww_z - 2\Omega u = -\frac{1}{\rho}P_z - g. && \text{(c)} \end{aligned} \tag{2c}$$

The boundary conditions for the fluid are given by

$$w = \eta_t + u\eta_x + v\eta_y \quad \text{on } y = \eta(x, y, t), \tag{2d}$$

$$P = P_0 \quad \text{on } y = \eta(x, y, t). \tag{2e}$$

Here η represents the free-surface and P_0 is the constant atmospheric pressure. The kinematic boundary condition on the surface simply states that all surface particles remain confined to the surface. Since we are interested in waves which are trapped in the equatorial region, we stipulate in the following that the wave surface profile decays in the latitudinal directions away from the equator. Finally, we assume the water to be infinitely deep, with the flow converging to a uniform current rapidly with depth, that is,

$$(u, v) \to (-c_0, 0) \quad \text{as } y \to -\infty. \tag{2f}$$

3. Lagrangian dynamics

In this section we define an exact solution of the β-plane governing equations (2). The solution represents steady waves travelling in the longitudinal direction, which have a constant speed of propagation $c > 0$, in the presence of a constant underlying current of strength c_0. We adopt the Lagrangian approach [24], whereby the Eulerian coordinates of fluid particles (x, y, z) are expressed as functions of the Lagrangian labelling variables $(q, r, s) \in (\mathbb{R}, (-\infty, r_0), \mathbb{R})$, and time t, as follows:

$$x = q - c_0 t - \frac{1}{k}e^{k[r-f(s)]}\sin[k(q - ct)], \tag{3a}$$

$$y = s, \tag{3b}$$

$$z = r + \frac{1}{k}e^{k[r-f(s)]}\cos[k(q - ct)], \tag{3c}$$

where $r_0 < 0$ and k is the wavenumber. The function $f(s)$ essentially determines the decay of the particle oscillation as it moves in the latitudinal direction away from the equator, and for the present construction we choose

$$f(s) = \frac{c\beta}{2\gamma}s^2, \tag{4}$$

where

$$\gamma = 2\Omega c_0 + g. \tag{5}$$

For notational convenience let us choose

$$\xi = k\left(r - f(s)\right), \qquad \theta = k(q - ct).$$

Then the Jacobian matrix of the transformation (3) is given by

$$\begin{pmatrix} \dfrac{\partial x}{\partial q} & \dfrac{\partial y}{\partial q} & \dfrac{\partial z}{\partial q} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} & \dfrac{\partial z}{\partial s} \\ \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} & \dfrac{\partial z}{\partial r} \end{pmatrix} = \begin{pmatrix} 1 - e^{\xi}\cos\theta & 0 & -e^{\xi}\sin\theta \\ f_s e^{\xi}\sin\theta & 1 & -f_s e^{\xi}\cos\theta \\ -e^{\xi}\sin\theta & 0 & 1 + e^{\xi}\cos\theta \end{pmatrix}. \tag{6}$$

The determinant of the Jacobian is $1 - e^{2\xi}$, which is time independent; thus it follows that the flow defined by (3) must be volume preserving, ensuring that (2b) holds in the Eulerian setting [24]. We further remark that, in order for the transformation (3) to be well-defined, and to furthermore ensure that our flow has the appropriate decay properties (in both the vertical and the latitudinal directions), we stipulate that

$$r - f(s) \le r_0 < 0. \tag{7}$$

We note that this relation forces the choice $c > 0$ for our flow. Bearing in mind that we are seeking trapped equatorial waves, we take $v \equiv 0$ throughout the fluid, and we calculate

$$u = \frac{Dx}{Dt} = ce^{\xi}\cos\theta - c_0, \qquad \frac{Du}{Dt} = kc^2e^{\xi}\sin\theta, \tag{8a}$$

$$v = \frac{Dy}{Dt} = 0, \qquad \frac{Dv}{Dt} = 0, \tag{8b}$$

$$w = \frac{Dz}{Dt} = ce^{\xi}\sin\theta, \qquad \frac{Dw}{Dt} = -kc^2e^{\xi}\cos\theta, \tag{8c}$$

where D/Dt is the material derivative. We can express (2c) as

$$\begin{aligned} &\frac{Du}{Dt} + 2\Omega w = -\frac{1}{\rho}P_x, \\ &\frac{Dv}{Dt} + \beta y u = -\frac{1}{\rho}P_y, \\ &\frac{Dw}{Dt} - 2\Omega u = -\frac{1}{\rho}P_z - g, \end{aligned}$$

and inserting the terms from (8) in this gives us

$$P_x = -\rho(kc^2e^{\xi}\sin\theta + 2\Omega ce^{\xi}\sin\theta), \tag{9a}$$

$$P_y = -\rho(\beta s[ce^{\xi}\cos\theta - c_0]), \tag{9b}$$

$$P_z = -\rho(-kc^2e^{\xi}\cos\theta - 2\Omega ce^{\xi}\cos\theta + \gamma). \tag{9c}$$