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Evaluation of absorption coefficient based on proper solution space

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ABSTRACT

This note is concerned with a new method for the evaluation of an unknown coefficient in a parabolic equation. The method starts with an initial guess for the unknown function and obtains corrections to the assumed value at every iteration. The updating step is the new feature of the present algorithm. The method treats the error field which has the appropriate (zero) Dirichlet boundary conditions. The algorithm shows good robustness to noise and can be used to obtain a good estimate of the unknown function. A number of numerical examples are used to show the applicability of the method.

1. Introduction

In this note, we introduce a new method for inverse coefficient evaluation of a parabolic system. Such problems arise in many applications including evaluation of permittivity distribution [1], ground water contamination [2], optical tomography [3] and heat transfer [18,19].

It is well-known that this problem is highly *ill-posed* [16], and various methods have been developed to deal with the ill-posedness that is associated with such problems. The literature on this problem is vast. Recent results on this particular problem include methods based on Quazi-Reversibility [17,4], minimization of a cost functional [5,6], multiple forward problem [7], elliptic system method [13], and methods based on inverse Sturm-Liouville problem [8,9].

The purpose of this note is to develop a new method for the evaluation of the absorption coefficient in one and two dimensions. In both cases we assume that the unknown function can be measured at the boundary, and for simplicity, we assume that it is equal to one. The proposed method was first developed for the inverse evaluation of a boundary condition in steady heat conduction problem [11]. It was also applied for the inverse evaluation of the initial condition for a parabolic system [14]. In Section 2, we present the algorithm. It assumes an initial value for the unknown function and obtains corrections to the assumed value. The new feature of the present algorithm is the updating stage which is presented in Section 3. In Section 4, we use a number of numerical examples to study the applicability of the method.

2. Problem statement and the identification algorithm

Let $\Omega = \{(t, x), x \in [0, 1], t \in [0, t^*)\}$ and consider a 1-D parabolic equation given by

$$u_t = u_{xx} + a(x)u, \quad (t, x) \in \Omega,$$
$$u(t, 0) = g_0(t), \quad u(t, 1) = g_1(t), \quad u(0, x) = u_0(x), \tag{1}$$

where $u(t,x)$ is the temperature, $a(x)$ is the absorption coefficient and Dirichlet boundary conditions are imposed. The unknown function is the absorption coefficient $a(x)$. This *ill-posed* problem is supplemented by the additional Neumann boundary conditions at the boundaries, i.e.

$$u_x|_{x=0} = f_0(t), \quad u_x|_{x=1} = f_1(t). \tag{2}$$

The inverse problem of interest here is to recover $a(x)$ based on the additional given conditions at the boundaries (2). For the purpose of inversion, we can define $u(t,x) = e^{v(t,x)}$ [10] (since $u(t,x) > 0$) and rewrite Eq. (1) according to

$$v_t = v_{xx} + v_x^2 + a(x), \quad (t, x) \in \Omega,$$
$$v(t, 0) = \ln(g_0(t)), \quad v(t, 1) = \ln(g_1(t)), \quad v(0, x) = \ln(u_0(x)). \tag{3}$$

The given data are transformed according to $v_x = u_x/u$ at the boundaries with $u > 0$ for all $t \in [0,t^*]$. This formulation is suitable because the unknown function is isolated in Eq. (3). The present algorithm is iterative and consists of three steps.

[1] Assume a value for the unknown coefficient $\hat{a}(x)$ and, using the given Dirichlet boundary conditions $g_0(t)$ and $g_1(t)$, obtain a background field satisfying the system

$$\hat{v}_t = \hat{v}_{xx} + \hat{v}_x^2 + \hat{a}(x), \quad (t, x) \in \Omega,$$
$$\hat{v}(t, 0) = \ln(g_0(t)), \quad \hat{v}(t, 1) = \ln(g_1(t)), \hat{v}(0, x) = \ln(u_0(x)). \tag{4}$$

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Notation

$u(t, x)$	Temperature
$a(x)$	Absorption coefficient
$g_0(t)$, $g_1(t)$	Temperature at the boundaries
$f_0(t)$, $f_1(t)$	Temperature gradients collected at the boundaries
$e(t, x)$	Error field
$\mathscr{B}_0$, $\mathscr{B}_1$	Differential operators describing the boundary conditions
$\mathbf{B}_0$, $\mathbf{B}_1$	Matrix representations for $\mathscr{B}_0$ and $\mathscr{B}_1$
$\epsilon_\ell(t, x)$	Elements in the proper solution space
$c_\ell(x)$, $\ell = 1, .., N$	A linearly independent set of functions
β, α	Positive parameters

[2] Subtract the background field from Eq. (3), and obtain the error field, $e(t, x) = v(t, x) - \hat{v}(t, x)$, given by

$$e_t = e_{xx} + v_x^2 - \hat{v}_x^2 + (a(x) - \hat{a}(x)), (t, x) \in \Omega,$$
$$e(t, 0) = e(t, 1) = e(0, x) = 0. \tag{5}$$

The error field is required to satisfy additional conditions. The given data is in the form of the gradient of the field at the boundaries, u_x

$$e_x|_{x=0} = \frac{f_0(t)}{g_0(t)} - \hat{v}_x|_{x=0} = \hat{f}_0(t), \quad e_x|_{x=1} = \frac{f_1(t)}{g_1(t)} - \hat{v}_x|_{x=1} = \hat{f}_1(t). \tag{6}$$

[3] Assume that the unknown function is related to the assumed value according to $a(x) = \hat{a}(x) + q(x)$, where $q(x)$ is still an unknown function. Use the additional boundary conditions in Eq. (6) and obtain the unknown correction $q(x)$ to the assumed value of $a(x)$. Update the assumed value, $\hat{a}(x)$, and go to step I.

The novel feature of the present method is the third step in the above algorithm which is the evaluation of the unknown absorption coefficient using a new method. The method was developed for an elliptic ill-posed heat conduction problem [11].

3. Proper solution space

The third step of the algorithm involves the identification of the correction to the assumed value of the absorption coefficient $a(x)$. We first linearize the quadratic terms in Eq. (5) according to

$$e_t = e_{xx} + v_x^2 - \hat{v}_x^2 + (a(x) - \hat{a}(x)) = e_{xx} + (v_x - \hat{v}_x)(v_x + \hat{v}_x) + q(x),$$
$$= e_{xx} + 2\hat{v}_x e_x + q(x). \tag{7}$$

To recover $q(x)$ for $x \in [0,1]$ we can proceed as follow. Consider a linearly independent set of functions $c_\ell(x)$, $\ell = 1,2,\ldots,N$ over $x \in [0,1]$ and assume that the unknown function $q(x)$ can be expressed as a linear combination of these functions, i.e. $q(x) \in \{c_1, c_2, \ldots, c_N\}$. We are assuming that it is possible to evaluate the unknown $a(x)$ at the boundaries, and as a result, we can assume that $q(0) = q(1) = 0$. Next, generate a set of functions that satisfy the error field equation with the known (zero) Dirichlet boundary condition, i.e.,

$$\epsilon_{\ell_t} = \epsilon_{\ell_{xx}} + 2\hat{v}_x \epsilon_{\ell_x} + c_\ell, \quad \epsilon_\ell(t, 0) = \epsilon_\ell(t, 1) = 0, \quad \epsilon_\ell(0, x) = 0. \tag{8}$$

Therefore, every function $\epsilon_\ell(t,x)$ satisfies the (zero) Dirichlet boundary conditions at $x = 0$ and $x = 1$, and zero initial condition. It is then possible to expand the actual (and unknown) error field $e(t,x)$ in the span of the space generated by $\epsilon_\ell(t,x)$, $\ell = 1,2,..,N$, according to

$$e(t, x) = \sum_{\ell=1}^{N} \tau_\ell \epsilon_\ell(t, x), \tag{9}$$

where the functions $\epsilon_\ell(t,x)$ are known, but the constants τ_ℓ are unknown. We next argue that the error field $e(t,x)$ must satisfy the gradient condition that is furnished by the measurements and are given in Eq. (6). The gradient conditions can be expressed by the operators $\mathscr{B}_0$ and $\mathscr{B}_1$. The error field is required to satisfy the conditions given by

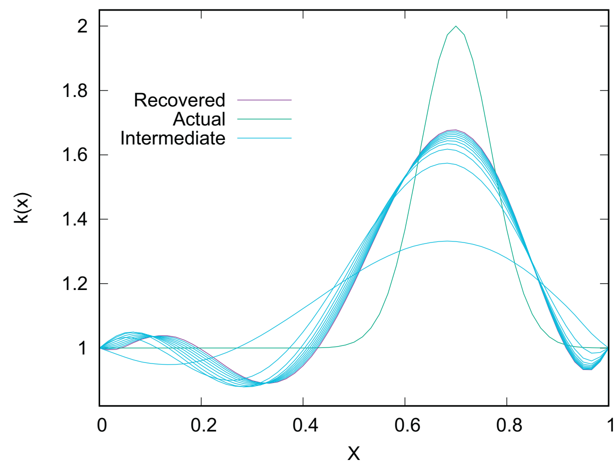

Fig. 1. The recovered absorption for the Example 1. The figure compares the final value to the actual function.

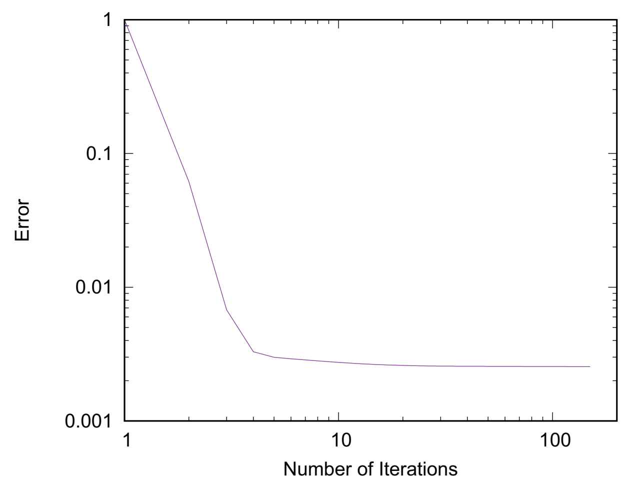

Fig. 2. The reduction in error for the Example 1 as a function of the number of iterations.

$$\mathscr{B}_0 e(t, 0) = \mathscr{B}_0 \sum_{\ell=1}^{N} \tau_\ell \epsilon_\ell(t, 0) = \sum_{\ell=1}^{N} \tau_\ell \mathscr{B}_0 \epsilon(t, 0) = \hat{f}_0(t), \tag{10}$$

$$\mathscr{B}_1 e(t, 1) = \mathscr{B}_1 \sum_{\ell=1}^{N} \tau_\ell \epsilon_\ell(t, 1) = \sum_{\ell=1}^{N} \tau_\ell \mathscr{B}_1 \epsilon(t, 1) = \hat{f}_1(t). \tag{11}$$

The above two equations can be used to obtain the unknown coefficients τ_ℓ for $\ell = 1,2,\ldots,N$. This step is presented in more details in the next section. Once the unknown coefficients are obtained the unknown $q(x)$ can be computed by substituting Eq. (9) in Eq. (7) which leads to

$$\sum_{\ell=1}^{N} \tau_\ell [\epsilon_{\ell_t} - \epsilon_{\ell_{xx}} - 2\hat{v}_x \epsilon_{\ell_x}] = q(x). \tag{12}$$

Using Eq. (8) and assuming an expansion for $q(x) = \sum_{\ell=1}^{N} \sigma_\ell c_\ell(x)$, and simplifying leads to