International Communications in Heat and Mass Transfer xxx (2015) xxx–xxx

Contents lists available at ScienceDirect

International Communications in Heat and Mass Transfer

journal homepage: www.elsevier.com/locate/ichmt

Traveling wave solutions to incompressible unsteady 2-D laminar flows with heat transfer boundary ☆

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A R T I C L E I N F O

Available online xxxx

Keywords:
Fluid dynamics
Navier–Stokes equations
Traveling wave method
Heat transfer
Laminar flow

A B S T R A C T

Analytical solutions play important roles in the understanding of fluid dynamics and heat transfer related problems. Some analytical solutions for incompressible steady/unsteady 2-D problems have been obtained in literature, but only a few of those are found under heat transfer conditions (which brings more complexities into the problem). This paper is focused on the analytical solutions to the basic problem of incompressible unsteady 2-D laminar flows with heat transfer. By using the traveling wave method, fluid dynamic governing equations are developed based on classical Navier–Stokes equations and can be reduced to ordinary differential equations, which provide reliable explanations to the 2-D fluid flows. In this study, a set of analytical solutions to incompressible unsteady 2-D laminar flows with heat transfer are obtained. The results show that both the velocity field and the temperature field take an exponential function form, or a polynomial function form, when traveling wave kind solution is assumed and compared in such fluid flow systems. In addition to heat transfer problem, the effects of boundary input parameters and their categorization and generalization of field forming or field evolutions are also obtained in this study. The current results are also compared with the results of Cai et al. (R. X. Cai, N. Zhang. International Journal of Heat and Mass Transfer, 2002, 45: 2623-2627) and others using different methods. It is found that the current method can cover the results and will also extend the fluid dynamic model into a much wider parameter ranges (and flow situations).

1. Introduction

Analytical solutions have always played important roles in the development of various fluid flow systems, which usually serve as fundamental basis for comparison of fluid dynamic nature and system evolution trends. For example, the analytical solutions of incompressible flow and constant coefficient heat conduction in early days have been the bases of fluid dynamics and heat transfer [1,2]. However, it is difficult to derive the analytical solutions of the governing equations with nonlinear terms, especially the complex governing equations (for fluid systems usually the Navier–Stokes equations are considered) with given initial and boundary conditions.

Although with the rapid development of computers and numerical methods, much research has focused on formulating efficient numerical methods to solve fluid dynamics and coupled heat transfer problems, the accuracy of these numerical solutions can only be ascertained by comparison with exact solutions or empirical/half-empirical correlations from experiments. Therefore, it is meaningful to find out some analytical solutions not only for the reason that they are found be able to describe the detailed behavior of the concerning system, but also that they can be used as benchmark solutions to check the accuracy, convergence and effectiveness of various numerical methods and solutions, and to improve various numerical methods such as their differencing schemes and grid generation skills [3–6].

For fluid dynamic systems, the well-known Navier–Stokes (N–S) equations, first introduced by Navier in 1821, and developed by Stokes in 1845, are the fundamental governing equations. For those two hundred years, many groups have tried to solve this problem. The work on the exact solutions of the Navier–Stokes equations has also accumulated in literature. However, due to the nonlinearity and complexity of Navier–Stokes equations, one can only give the solutions to very limited/simplified cases. Indeed there only exist a small number of exact solutions in literature. In the paper of Wang [7,8], one can find the historical reviews of the trials and solutions up to year 1991. And in the most recent years, super computers have made it possible to numerically solve the Navier–Stokes equations and the accuracy of the results can be compared with an exact solution. Thus, the exact solutions are very important as a test to verify numerical or empirical methods for complex flow problems. In the recent twenty or thirty years, major developments of the exact solutions of the Navier–Stokes equations can be

☆ Communicated by W.J. Minkowycz.

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http://dx.doi.org/10.1016/j.icheatmasstransfer.2015.05.006
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Nomenclature

α	thermal diffusivity, [m^2/s]
υ	kinematic viscosity, [m^2/s]
C_p	specific heat, [J/(kg·K)]
ρ	density, [kg/m^3]
g	gravitational acceleration, [m/s^2]
θ	temperature, [K]
p	pressure, [Pa]
u,w	velocity components in x, z directions, [m/s]
t	time, [s]
u_0	initial velocity, [m/s]
p_0	initial pressure, [Pa]
x	abscissa, [m]
z	third coordinate, [m]
ξ	transformation $\xi = ax + bz + ct$, [m]
a,b	constants, $a^2 + b^2 \neq 0$
c	wave speed, [m/s]
m_i, M_j	constants used during the integration of equations

found in Chandna and Oku-Ukpong [9], Profilo et al. [10], Siddique [11], Venkatalaxmi et al. [12], Warsi [13], Naeem and Jamil [14], Naeem and Younus [15] and others [16].

The major methods used in solving the Navier–Stokes equations are variable separation and equation transformation methods; the equation transformation method then includes potential function assumption and parameter transformation. The basic process is to change/separate the equations to reduce the problem to the class of nonlinear ordinary differential equations for the first step [17]. Indeed, variable separation has been widely used in solving complex equations. Potential functions are also often used in similar methods [16,18]. This method has to assume the independence relations between the variables, making it difficult to justify the fundamentals of the solving process and the results obtained. Besides the results mentioned in review paper of Wang [7,8], in recent years representative ones can be found by many groups. For example, Al-Mdallal [19] utilized the canonical transformation with complex coefficients; the Navier–Stokes equations were reduced to a linear partial differential equations that can be solved by using separation of variables. According to the type of the canonical transformation constants (real or complex), Al-Mdallal [19] got different types of exact solutions. By using variable separation method with addition, Cai and Zhang. [20] obtained some exact solutions to incompressible unsteady 2-D laminar flow with heat transfer, neglecting gravity, radiation and internal heat source. By this method, Cai discussed a series of problems described by Navier–Stokes equations and found a set of useful solutions to some basic flow situations [21,22].

Variable or equation transformation methods are also generally seen in literatures. For example, Nugroho et al. [17,23] proposed a potential function and transformed coordinate to alter the three-dimensional incompressible Navier–Stokes equations into simpler forms. Furthermore, a special class of solutions to the three-dimensional incompressible Navier–Stokes equations was obtained by dropping the pressure gradient and it was found that constant pressure gradient will produce similar solutions to that of a zero pressure gradient. The authors also proposed another potential function and transformed coordinate, which has a nontrivial relation with respect to time, and a general functional form of static pressure was applied. Fang et al. [24] have investigated the steady momentum and heat transfer of a viscous fluid flow over a stretching/shrinking sheet, and have presented new exact solutions for the Navier–Stokes equations. These solutions provide a more general formulation including the linear stretching and shrinking wall problems as well as the asymptotic suction velocity profiles over a variety of situations.

As discussed, one case was the transformation of the Navier–Stokes equations to the Schrödinger equation, performed by application of the Riccati equation [25] and to achieve much simpler forms. This has good prospects since the Schrödinger equation is linear and has well defined solutions. The method of Lie group theory was also applied in order to transform the original partial differential equations into ordinary differential systems [26]. The same route was followed by Meleshko [27] and by Thailert [28], in transforming the Navier–Stokes equations to solvable linear systems.

The current study is one continued trial of solving channel flow with heat transfer by using the transformation method. The current study is focused on the transforming of partial differential equations into tractable ordinary differential equations or some particular partial differential equations. Traveling wave method belongs to this family; due to the application of the transform $\xi = \sum a_i x_i$, partial differential equations can be reduced to tractable ordinary differential equations, where ξ is a variable of ordinary differential equations and it is called a phase of the wave, $x_1, \ldots, x_n$ are independent variables of the partial differential equations, and $a_1, \ldots, a_n$ are arbitrary constants.

In this paper, transformation method and traveling wave solution are used to solve the incompressible unsteady 2-D laminar flow with boundary heat transfer problem. The current study takes the simplified 2-D laminar flow conditions, where the instability shear flow with wave transportation happens. Such phenomena indeed are fundamental and critical during the formation of shear flow and the establishment of viscous and thermal boundary layers. The basic mathematical model are carefully established and tested with several general cases. It is found that the current model is capable of covering such benchmark cases and can provide new information on the parameter evolutions of Navier–Stokes governed fluid dynamic systems. The following parts of this paper are arranged as follows: In Section 2, the government equations of the incompressible unsteady 2-D laminar flow with heat transfer are presented. In Section 3, two kinds of traveling wave solutions are obtained under various conditions. For the first one, the velocity and temperature fields mainly depend on exponential functions, while for the second one, they depend on polynomial functions. In Section 4, the benchmark case results are analyzed and explained in detail. Further, the physical descriptions of these solutions are explained and some solutions with certain conditions are shown. A short conclusion is given in Section 5.

2. Basic model description and governing equations

2.1. Physical model and basic Navier–Stokes equations

In this paper, the general governing equations of the incompressible unsteady 2-D laminar flow with heat transfer is considered. The governing equations consisted of continuity equation, incompressible fluid Navier–Stokes equation and thermal conservation equation, which are written as follows (neglecting radiation and internal heat source; z-direction is opposite to gravitation) [20]:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \upsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \upsilon\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right) - g \tag{3}$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + w\frac{\partial \theta}{\partial z} = \alpha\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial z^2}\right) + \frac{\upsilon}{C_p}\left[\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2\right] \tag{4}$$

Please cite this article as: Y. Zhao, et al., Traveling wave solutions to incompressible unsteady 2-D laminar flows with heat transfer boundary, Int. Commun. Heat Mass Transf. (2015), http://dx.doi.org/10.1016/j.icheatmasstransfer.2015.05.006