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Q1
Identification of a time-dependent bio-heat blood perfusion coefficient☆

Q2
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ABSTRACT

In the paper the identification of the time-dependent blood perfusion coefficient is formulated as an inverse problem. The bio-heat conduction problem is transformed into the classical heat conduction problem. Then the transformed inverse problem is solved using the method of fundamental solutions together with the Tikhonov regularization. Some numerical results are presented in order to demonstrate the accuracy and the stability of the proposed meshless numerical algorithm.

1. Introduction

The mathematical description of the relation between the tissue temperature and the arterial blood perfusion has received a quite extensive attention over the years in the literature. Several different models for heat transfer in living tissues have been proposed, e.g. Pennes' bioheat transfer model [1], which is the most commonly used, Wulff continuum model [2], Klinger continuum model [3], continuum model of Holmes and Chen [4] and Weinbaum–Jiji bioheat model [5]. Q3

A number of standard numerical methods have been used in order to solve the heat transfer in living tissues, e.g. the finite element method [6], the finite difference method [7], the boundary element method (BEM) [8], the dual reciprocity BEM [9], the Trefftz finite element method [10] or the Monte Carlo method [11]. Recently, some papers investigated the application of meshless methods for solving the bio-heat equation, e.g. Yousefi in Ref.[12] used the Bernstein–Galerkin method for finding the time-dependent blood perfusion coefficient from internal tissue temperature measurements, Cao et al. in Ref.[13] used the method of fundamental solutions (MFS) in combination with radial basis functions for analysing thermal behavior of skin tissues, whilst the radial basis collocation method was used by Jamil and Ng in Ref.[14] to predict the temperature inside biological tissues.

In the present paper, the identification of the time-dependent blood perfusion coefficient in bioheat conduction is formulated as an inverse problem, which is then solved numerically using the MFS. In order to ensure stability of the solution for noisy data the Tikhonov regularization is employed.

2. Mathematical formulation

The bioheat equation proposed by Pennes [1] can be written as:

$$k_t \Delta \mathrm{T} - w_b C_b \left(\mathrm{T} - \mathrm{T_a}\right) + S = \rho_t c_t \frac{\partial \mathrm{T}}{\partial t^*}, \tag{1}$$

where k_t is the thermal conductivity of the tissue, T is the temperature of the tissue, w_b is the blood perfusion rate, C_b is the heat capacity of blood, T_a is the temperature of the arterial blood, S is a volumetric source related to heat generation due to metabolism and heat deposition, ρ_t and c_t are, respectively, the density and specific heat of the tissue, and $t*$ is the time.

In one-dimension, Eq. (1) takes the following dimensionless form:

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) - q(t)\,u(x,t) + f_S, \quad (x,t) \in (0,1) \times (0,T], \tag{2}$$

where

$$x = \frac{x^*}{L}, \quad t = \frac{t^* k_t}{L^2 \rho_t c_t}, \quad u = \frac{k_t\left(\mathrm{T} - \mathrm{T_a}\right)}{S_0 L^2}, \quad f_S = \frac{S}{S_0}, \quad q = \frac{w_b c_b L^2}{k_t}. \tag{3}$$

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Neglecting, for simplicity, the dimensionless heat source f_S we consider the inverse problem of finding the temperature $u(x,t)$ and the time-dependent blood perfusion coefficient $q(t)$ satisfying the bio-heat conduction equation:

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) - q(t)\,u(x,t), \quad (x,t) \in (0,1)\times(0,T] \tag{4}$$

subject to the initial condition

$$u(x,0) = u_0(x), \quad x \in [0,1], \tag{5}$$

the Robin boundary condition at $x = 1$, namely

$$\alpha u(1,t) + \beta \frac{\partial u}{\partial x}(1,t) = g(t), \quad t \in (0,T], \tag{6}$$

the Dirichlet boundary condition at $x = 0$, namely

$$u(0,t) = f(t), \quad t \in (0,T], \tag{7}$$

and the Neumann boundary condition at $x = 0$, namely

$$-\frac{\partial u}{\partial x}(0,t) = h(t), \quad t \in (0,T]. \tag{8}$$

In Eq. (6), we typically choose $\alpha = 1, \beta = 0$, i.e. the Dirichlet boundary condition

$$u(1,t) = g_0(t), \quad t \in (0,T], \tag{9}$$

or $\alpha = 0, \beta = 1$, i.e. the Neumann boundary condition

$$\frac{\partial u}{\partial x}(1,t) = g_1(t), \quad t \in (0,T]. \tag{10}$$

We also assume the continuity compatibility conditions

$$u_0(0) = f(0), -u_0'(0) = h(0), \quad u_0(1) = g_0(0), \quad u_0'(1) = g_1(1). \tag{11}$$

Instead of the heat flux Condition (8), we can have the temperature measurement at an interior point $X_0 \in (0,1)$ giving the data

$$u(X_0,t) = \phi(t), \quad t \in (0,T] \tag{12}$$

satisfying the continuity condition

$$\phi(0) = u_0(X_0). \tag{13}$$

Another alternative to Eq. (8) is to prescribe the total mass measurement

$$\int_0^1 u(x,t)\,dx = E(t), \quad t \in (0,T] \tag{14}$$

satisfying the continuity condition

$$E(0) = \int_0^1 u_0(x)\,dx. \tag{15}$$

Under certain additional conditions, the uniqueness of the inverse Problems (4), (5), (7), (9) and (12) or (14)has been proved by Prilepko and Solov'ev [15] and Lin [16], respectively. Merever, both these inverse problems, also in multi-dimensions, have been investigated by Cannon et al. [17]. Reduction of the inverse Problems (4), (5), (7), (8) and (10) to a well-posed Volterra integral equation of the second kind has been proved in Refs.[18, 19] and with multi-dimensional extensions given in Ref. [20].

In this study, we start with the theoretical analysis for solving the alternative inverse Problems (4), (5), (7)–(9).

We first observe that employing the transformations

$$r(t) = \exp\left(\int_0^t q(\tau)\,d\tau\right), \quad v(x,t) = r(t)\,u(x,t) \tag{16}$$

recasts the Problems (4), (5), (7)–(9) into the form

$$\frac{\partial v}{\partial t}(x,t) = \frac{\partial^2 v}{\partial x^2}(x,t), \quad (x,t) \in (0,1)\times(0,T], \tag{17}$$

$$v(x,0) = u_0(x), \quad x \in [0,1], \tag{18}$$

$$v(0,t) = r(t)f(t), \quad v(1,t) = r(t)\,g_0(t), \quad t \in (0,T], \tag{19}$$

$$-\frac{\partial v}{\partial x}(0,t) = r(t)\,h(t), \quad t \in (0,T]. \tag{20}$$

Imposing also the first three continuity conditions in Eq. (11) leads to

$$r(0) = 1. \tag{21}$$

Thus using Eq. (16), the bio-heat Eq. (4) transforms into the heat Eq. (17). Applying the Green formula to the Problems (17)(20), a system is obtained consisting of two integral equations

$$\begin{aligned}\frac{1}{2}r(t)f(t) = &\int_0^t r(\tau)\left[G(0,t;0,\tau)\,h(\tau) + \frac{\partial G}{\partial \xi}(0,t;0,\tau)f(\tau)\right.\\ &\left.-\frac{\partial G}{\partial \xi}(0,t;1,\tau)\,g_0(\tau)\right]d\tau + \int_0^t r(\tau)\,g_1(\tau)\,G(0,t;1,\tau)\,d\tau\\ &+\int_0^1 G(0,t;y,0)\,u_0(y)\,dy, \quad t \in (0,T],\end{aligned} \tag{22}$$

$$\begin{aligned}\frac{1}{2}r(t)\,g_0(t) = &\int_0^t r(\tau)\left[G(1,t;0,\tau)\,h(\tau) + \frac{\partial G}{\partial \xi}(1,t;0,\tau)f(\tau)\right.\\ &\left.-\frac{\partial G}{\partial \xi}(1,t;1,\tau)\,g_0(\tau)\right]d\tau + \int_0^t r(\tau)\,g_1(\tau)\,G(1,t;1,\tau)\,d\tau\\ &+\int_0^1 G(1,t;y,0)\,u_0(y)\,dy, \quad t \in (0,T],\end{aligned} \tag{23}$$

where G is the fundamental solution of the one-dimensional time-dependent heat equation

$$G(x,t;\xi,\tau) = \frac{H(t-\tau)}{\sqrt{4\pi(t-\tau)}}\exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right), \tag{24}$$

with H as the Heaviside function.

Using Eq. (24), Eqs. (22) and (23) can be rewritten as a linear system of Volterra integral equations of the second kind, namely,

$$\begin{aligned}\sqrt{\pi}r(t)f(t) = &\int_0^t \frac{r(\tau)}{\sqrt{t-\tau}}\left[h(\tau)+g_1(\tau)+\frac{g_0(\tau)}{2(t-\tau)}\exp\left(-\frac{1}{4(t-\tau)}\right)\right]d\tau\\ &+\frac{1}{\sqrt{t}}\int_0^1 u_0(y)\exp\left(-\frac{y^2}{4t}\right)dy, \quad t \in (0,T],\end{aligned} \tag{25}$$