International Journal of Heat and Fluid Flow 50 (2014) 145–159

Contents lists available at ScienceDirect

International Journal of Heat and Fluid Flow

journal homepage: www.elsevier.com/locate/ijhff

A proper orthogonal decomposition method for nonlinear flows with deforming meshes

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ARTICLE INFO

Article history:
Received 11 March 2014
Received in revised form 31 May 2014
Accepted 4 July 2014
Available online 1 August 2014

Keywords:
Proper orthogonal decomposition
Reduced-order modeling
Computational fluid dynamics
Deforming mesh

ABSTRACT

This paper presents a proper orthogonal decomposition (POD) method that uses dynamic basis functions. The dynamic functions are of a prescribed form and do not explicitly depend on time but rather on parameters associated with flow unsteadiness. This POD method has been developed for modeling nonlinear flows with deforming meshes but can also be applied to fixed meshes. The method is illustrated for subsonic and transonic flows in channels with fixed and deforming meshes. This method properly captured flow nonlinearities and shock motion for cases in which the classical POD method failed.

1. Introduction

Despite continuous advances in computer hardware, the computational cost of high-fidelity computational fluid dynamics simulations remains a limiting factor for many science- and engineering-relevant problems. A typical example of numerical simulations that require large computational resources is aeroelasticity, where unsteadiness of the flow and temporal variation of the mesh can be a computational burden.

Reduced-order modeling based on proper orthogonal decomposition (POD) has proven to be a successful method for reducing the computational time, while providing high-fidelity results for a wide range of applications covering transport phenomena and structural dynamics (Dowell and Hall, 2001). Through model reduction, dominant spatial modes are used to describe the flow. The nonlinear partial differential equations can then be reduced to ordinary differential equations from which the time coefficients that weight the spatial modes are calculated.

Proper orthogonal decomposition is a method through which snapshots of the flow obtained from the full-order model (FOM) are used to extract the optimal set of spatially dependent basis functions (Holmes et al., 1996). The large set of partial differential equations is then projected onto the basis functions, resulting in a much smaller set of ordinary differential equations.

Reviews of POD-based reduced-order models (ROMs) have been presented in (Dowell and Tang, 2003; Lucia et al., 2004; Barone and Payne, 2005; Noack et al., 2011). In the last decade, three main research directions were explored for POD-based ROMs: (i) improving the prediction of off-reference conditions, (ii) improving performance, and (iii) modeling moving/deforming meshes.

Proposed modifications to the POD basis functions to account for off-reference conditions include direct interpolation, enriching the snapshot database (Schmit and Glauser, 2004), interpolation using subspace angles (Lieu and Lesoinne, 2004; Lieu et al., 2006; Lieu and Farhat, 2007) or a tangent space to a Grassmann manifold (Amsallem and Farhat, 2008; Amsallem, 2010; Freno et al., 2013), sensitivity analysis using parametric derivatives (Hay et al., 2008, 2010), and using actuation modes (Kasnakoğlu et al., 2008; Bourguet et al., 2011). Some of these methods are reviewed in Vetrano et al. (2011).

To improve performance for compressible flows, the use of physically or numerically sensible inner products has been suggested to better account for dynamically significant variables (Rowley et al., 2004) and to improve ROM stability (Barone et al., 2009). For multiphase flows, Brenner et al. (2012) showed that treating field variables separately when assembling the autocorrelation matrix, which yields the POD basis functions, produces greater error than using a coupled approach. To solve flows with discontinuities, an augmented POD method (Brenner et al., 2010) was developed using mathematical morphology. Several acceleration techniques were proposed in Cizmas et al. (2008).

The modeling of moving/deforming meshes has been primarily motivated by aeroelastic applications, which are notorious for requiring large computational resources. POD has been used in linear (Hall et al., 2000; Thomas et al., 2003; Lieu et al., 2006;

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http://dx.doi.org/10.1016/j.ijheatfluidflow.2014.07.001
0142-727X/

Amsallem and Farhat, 2008; Bui-Thanh et al., 2008) and nonlinear aeroelastic simulations (Anttonen, 2001; Lewin and Haj-Hariri, 2005; Anttonen et al., 2005; Placzek et al., 2011). One of the primary challenges associated with nonlinear aeroelastic simulations is the motion of the mesh, particularly when it is deformed. Spatial and temporal integration no longer commute when the mesh varies in time. However, if the mesh is deformed in a topologically consistent manner, the integrals can commute if a computational index-based domain is used.

Anttonen (2001) and Anttonen et al. (2003, 2005) proposed using different sets of index-based basis functions associated with different deformations; however, discontinuously changing basis functions with respect to time reduces the solution fidelity. Additionally, several sets of basis functions are required to yield a robust model, and a matching algorithm is necessary to determine the most appropriate set.

Liberge and Hamdouni (2010) used interpolation by treating the fluid–structure domain as a multiphase flow. In addition to requiring interpolation, modifications to the boundary conditions are required. Lewin and Haj-Hariri (2005) modeled the incompressible Navier–Stokes equations by using the reference frame of the moving airfoil to exploit the simplified boundary conditions that arise from incompressible viscous flow. Placzek et al. (2011) modeled compressible flow for rigid-body motions. These approaches do not address mesh deformation.

Bourguet et al. (2011) developed an approach to model transonic flows around an airfoil submitted to small deformations. The domain was deformed fictitiously through a Hadamard formulation of the compressible Navier–Stokes equations. To account for the deformation, the boundary conditions about the airfoil were modified.

This paper presents a new, index-based method that uses a dynamic average and dynamic basis functions to model compressible flow using a deforming mesh. There is no need for interpolation or modification of the boundary conditions. These dynamic functions vary continuously with respect to parameters associated with the flow unsteadiness and/or mesh deformation, and they are optimal, subject to the prescribed form. Furthermore, one set of basis functions is used, and a matching algorithm is unnecessary.

The derivation of the dynamic average and basis functions is presented in Section 2, and the flow solver is described in Section 3. In Section 4, results are shown for subsonic and transonic flows with fixed and deforming meshes. Comparisons are made between the full-order model and the reduced-order model using static and dynamic functions. The results are discussed in Section 5, and conclusions are presented in Section 6.

2. Proper orthogonal decomposition

Proper orthogonal decomposition is a method through which an optimal set of orthogonal spatial basis functions is extracted from a set of data, from which the mean has typically been subtracted. The spatial basis functions are linearly combined using time-dependent coefficients to form a reduced-order model:

$$\mathbf{U}(\mathbf{x},t) \approx \overline{\mathbf{U}}(\mathbf{x}) + \sum_{j=1}^{m} a_j(t)\boldsymbol{\varphi}_j(\mathbf{x}). \tag{1}$$

In (1), $\overline{\mathbf{U}}(\mathbf{x})$ is the time average, $a_j(t)$ are the time coefficients, and $\boldsymbol{\varphi}_j(\mathbf{x})$ are the basis functions. Through reduced-order modeling, the partial differential equations are reduced to a system of ordinary differential equations.

In this paper, proposed modifications to POD include replacing the static average and static basis functions with a dynamic average and dynamic basis functions. The dynamic average and dynamic basis functions do not explicitly depend on time but rather on parameters associated with the flow unsteadiness and/or mesh deformation.

The dynamic functions used in this paper take the form

$$f(\mathbf{x};\gamma,\dot{\gamma}) = f_0(\mathbf{x}) + \gamma f_1(\mathbf{x}) + \dot{\gamma} f_2(\mathbf{x});$$

however, this function can be trivially extended to account for additional parameters, higher derivatives, and/or greater polynomial degree, provided all parameters, derivatives, and multiples thereof are linearly independent.

The first subsection outlines the procedure for determining the static basis functions (Sirovich, 1987; Holmes et al., 1996), and the remaining subsections show how the optimal dynamic average and dynamic basis functions of the prescribed form are computed.

2.1. Standard approach

A more general framework for the traditional approach to POD is presented to facilitate the extensions proposed later in this section. Conventionally, after subtracting the time average, $\overline{\mathbf{U}}$, from the snapshots, $\mathbf{U}, \mathbf{U}' \equiv \mathbf{U} - \overline{\mathbf{U}}$ is approximated by

$$\mathbf{U}'(\mathbf{x},t) \approx \sum_{j=1}^{m} a_j(t)\boldsymbol{\varphi}_j(\mathbf{x}),$$

where $a_j(t) = (\mathbf{U}'(\mathbf{x},t), \boldsymbol{\varphi}_j(\mathbf{x}))/(\boldsymbol{\varphi}_j(\mathbf{x}), \boldsymbol{\varphi}_j(\mathbf{x}))$, and $(\cdot,\cdot)$ is the inner product. The basis functions have been presumed mutually orthogonal to more efficiently span the subspace. $\mathbf{U}'$ is equal to the sum of the approximation obtained from the projection onto the basis and the error:

$$\mathbf{U}' = \sum_{j=1}^{m} \frac{\left(\mathbf{U}', \boldsymbol{\varphi}_j\right)}{\left(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_j\right)}\boldsymbol{\varphi}_j + \left(\mathbf{U}' - \sum_{j=1}^{m} \frac{\left(\mathbf{U}', \boldsymbol{\varphi}_j\right)}{\left(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_j\right)}\boldsymbol{\varphi}_j\right).$$

Since the error is orthogonal to the approximation, the Pythagorean theorem holds, and

$$\|\mathbf{U}'\|^2 = \left\| \sum_{j=1}^{m} \frac{\left(\mathbf{U}', \boldsymbol{\varphi}_j\right)}{\left(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_j\right)}\boldsymbol{\varphi}_j \right\|^2 + \left\| \mathbf{U}' - \sum_{j=1}^{m} \frac{\left(\mathbf{U}', \boldsymbol{\varphi}_j\right)}{\left(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_j\right)}\boldsymbol{\varphi}_j \right\|^2,$$

where $\|\cdot\|$ is the L^2-norm. Consequently, minimizing the time-averaged error is equivalent to maximizing the time-averaged approximation. Due to the orthogonality assumption, the time-averaged square of the norm of the approximation can be simplified to

$$\left\langle \left\| \sum_{j=1}^{m} \frac{\left(\mathbf{U}', \boldsymbol{\varphi}_j\right)}{\left(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_j\right)}\boldsymbol{\varphi}_j \right\|^2 \right\rangle = \left\langle \sum_{j=1}^{m} \frac{\left(\mathbf{U}', \boldsymbol{\varphi}_j\right)^2}{\left(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_j\right)} \right\rangle,$$

as shown in Appendix A. $\langle\cdot\rangle$ denotes the time average.

The norm of the approximation is maximized by determining the optimal basis functions that maximize the functional

$$J[\boldsymbol{\varphi}] \equiv \left\langle \frac{\left(\mathbf{U}', \boldsymbol{\varphi}\right)^2}{\left(\boldsymbol{\varphi}, \boldsymbol{\varphi}\right)} \right\rangle, \tag{2}$$

where the subscript j has been removed for convenience. Using the notation $\widehat{\mathbf{A}}(t) \equiv \mathbf{U}'(\mathbf{x},t) \otimes \mathbf{U}'(\mathbf{x},t)$ yields $\left(\mathbf{U}', \boldsymbol{\varphi}\right)^2 \equiv \boldsymbol{\varphi}^T\widehat{\mathbf{A}}\boldsymbol{\varphi}$, so that (2) becomes

$$J[\boldsymbol{\varphi}] \equiv \left\langle \frac{\boldsymbol{\varphi}^T\widehat{\mathbf{A}}(t)\boldsymbol{\varphi}}{\left(\boldsymbol{\varphi}, \boldsymbol{\varphi}\right)} \right\rangle. \tag{3}$$

As shown in Appendix B, (3) is extremized when

$$\left\langle \frac{\widehat{\mathbf{A}}(t)\boldsymbol{\varphi}}{\left(\boldsymbol{\varphi}, \boldsymbol{\varphi}\right)} - \frac{\left(\boldsymbol{\varphi}^T\widehat{\mathbf{A}}(t)\boldsymbol{\varphi}\right)\boldsymbol{\varphi}}{\left(\boldsymbol{\varphi}, \boldsymbol{\varphi}\right)^2} \right\rangle = \mathbf{0}.$$