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Element differential method for solving transient heat conduction problems

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ABSTRACT

In this paper, a new numerical method, Element Differential Method (EDM), is developed for solving transient heat conduction problems with variable conductivity. The key point of this method is based on the direct differentiation of shape functions of isoparametric elements used to evaluate the geometry and physical variables. A new collocation method is proposed for establishing the system of equations, in which the governing differential equation is collocated at nodes inside elements, and the flux equilibrium equation is collocated at interface nodes between elements and outer surface nodes of the problem. Attributed to the use of the Lagrange elements that can guarantee the variation of physical variables consistent through all elemental nodes, EDM has higher stability than the traditional collocation method. The other main characteristics of EDM are that no variational principle or a control volume are required to set up the system of equations and no integrals are included to form the coefficients of the system. Based on the implicit backward differentiation scheme, an unconditionally stable and non-oscillatory time marching solution scheme is developed for solving the time-dependent system equations. Numerical examples are presented to demonstrate the accuracy and efficiency of the proposed method.

1. Introduction

Transient heat transfer analysis is of great importance in many practical engineering areas [1–3]. The solution techniques to transient heat conduction problems are mainly based on analytical and numerical methods. The analytic method is accurate, but only available for isotropic homogeneous problems with simple geometries and boundary conditions, and, therefore, has limited application. The numerical method is a very flexible and robust way to solve complex heat conduction problems. The commonly used numerical methods can be classified into four types: the finite element method (FEM) [4,5], the finite volume method (FVM) [6,7], the boundary element method (BEM) [2,8] and meshless methods [9,10]. Compared to FEM and FVM, BEM is very robust for solving the heat conduction problem, since it only needs the discretization of the problem boundary into elements, rather than the whole domain, thus reducing the dimension of the problem by one [8]. However, BEM faces a critical challenge when solving non-linear [3,11], non-homogeneous [2] and transient [12–15] problems, since usually there are domain integrals concerned in the resulting integral equations, thus making BEM lose its unique advantage of boundary only discretization. To avoid this deficiency, some methods of transforming domain integrals into equivalent boundary integrals are developed and have been frequently used. In these methods, the dual reciprocity method (DRM) developed by Brebbia [8,16] is extensively employed. However, DRM requires particular solutions to basis functions, which restricts its application to complicated problems. Recently, a new transformation method, the radial integration method (RIM), has been proposed by Gao [17,18], which not only can transform any complicated domain integrals to the boundary in a unified way without using particular solutions, but also can remove various singularities appearing in the domain integrals. For solving transient heat conduction problems, Yang and Gao [19] developed a new boundary element analysis approach based on RIM, in which RIM is used to clear up the domain integral associated with the time derivative of temperatures, and the radial integral is evaluated numerically. Then a new and simple boundary-domain integral equation is presented for solving nonlinear [20] and transient nonlinear [21] heat conduction problems with temperature-dependent conductivity of materials. By considering that the numerical evaluation of the radial integrals is very time-consuming, Yang and Gao [22–24] developed a set of new analytical expressions for evaluating radial integrals appearing in the computation of several kinds of variable coefficient problems using the radial integration boundary element

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method (RIBEM). Through employ of the derived analytical expressions, the computational efficiency can be increased considerably.

Recently, a new robust method, element differential method (EDM) [25,26], is proposed for solving general heat conduction problems [25], elastic mechanics and thermal stress problems [26] based on the use of isoparametric elements as used in the standard FEM [4]. A set of explicit formulations of computing the first and second order spatial derivatives are derived for 2D and 3D problems. These formulations are presented for shape functions of elements and therefore can be used to any physical variables' differentiation. Since EDM can use high order isoparametric elements to compute the spatial derivatives, the computational accuracy in heat flux is higher than the frequently used FVM method. In this paper, a new type of element differential method is developed for solving transient heat conduction problems with variable conductivity for the first time. Without the complexity of solving the transient heat conduction BEM as before, the most important characteristic of the proposed method is that the derived spatial derivatives can be directly substituted into the governing equations and the heat flux equilibrium equations to form the final system of algebraic equations. So EDM is very easy to be coded in dealing with transient heat conduction engineering problems with complicated governing equations and boundary conditions.

2. Governing equations for non-homogeneous transient heat conduction problems

The governing equation for transient heat conduction problems in isotropic non-homogeneous media can be expressed as follows:

$$\frac{\partial}{\partial x_i}(k_{ij}(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial x_j}) + Q(\mathbf{x}) = \rho c\left(\frac{\partial T(\mathbf{x},t)}{\partial t}\right) \qquad t \geqslant t_0, \mathbf{x} \in \Omega \tag{1}$$

The boundary conditions of the problem are

$$T(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \Gamma_1 \quad \text{for Dirichlet boundary condition} \tag{2a}$$

$$q_i(\mathbf{x}) = -k_{ij}\frac{\partial T(\mathbf{x})}{\partial x_j} = g_i(\mathbf{x}),\ \mathbf{x} \in \Gamma_2 \quad \text{for Neumann boundary condition} \tag{2b}$$

$$q_{\mathbf{n}}(\mathbf{x}) = -k_{ij}\frac{\partial T(\mathbf{x})}{\partial x_j}n_i = h(x)(T(\mathbf{x}) - T_\infty),\ \mathbf{x} \in \Gamma_3 \quad \text{for Robin boundary condition} \tag{2c}$$

where x_i is the i th component of the spatial coordinates at point $\mathbf{x} = (x_1, x_2, \ldots, x_{ndim})$, *ndim* the dimension of problems, $k(\mathbf{x})$ the thermal conductivity, $T(\mathbf{x})$ the temperature, and $Q(\mathbf{x})$ the heat-generation rate.

3. Derivatives of elemental shape functions with respect to global coordinates

Any variables varying over an isoparametric element can be represented in terms of their nodal values of the element [4]. For example, the spatial coordinates and temperature can be interpolated as

$$x_i = N_\alpha x_i^\alpha, \quad T = N_\alpha T^\alpha \tag{3}$$

where x_i^α, N_α and T^α are the values of coordinates, shape function and temperature at node α, respectively, and the repeated index α represents the summation over all nodes. To numerically compute the partial derivatives appearing in the governing Eq. (1) and boundary conditions (2), the analytical expressions for the first and second partial derivatives need to be derived. From Eq. (3) it follows that

$$\frac{\partial T}{\partial x_i} = \frac{\partial N_\alpha}{\partial x_i}T^\alpha, \quad \frac{\partial^2 T}{\partial x_i \partial x_j} = \frac{\partial^2 N_\alpha}{\partial x_i \partial x_j}T^\alpha \tag{4}$$

It can be seen that N_α are the explicit functions of intrinsic coordinates $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_{ndim})$, thus

$$\frac{\partial T}{\partial x_i} = \frac{\partial N_\alpha}{\partial x_i}T^\alpha = \frac{\partial N_\alpha}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_i}T^\alpha = \frac{\partial N_\alpha}{\partial \xi_j}[J]_{ij}^{-1}T^\alpha \tag{5}$$

$$\frac{\partial^2 T}{\partial x_i \partial x_k} = \frac{\partial}{\partial x_k}\frac{\partial N_\alpha}{\partial \xi_j}[J]_{ij}^{-1}T^\alpha = \left([J]_{ij}^{-1}\frac{\partial^2 N_\alpha}{\partial \xi_j \partial \xi_l} + \frac{\partial [J]_{ij}^{-1}}{\partial \xi_l}\frac{\partial N_\alpha}{\partial \xi_j}\right)[J]_{kl}^{-1}T^\alpha \tag{6}$$

where $[J] = [\partial \mathbf{x}/\partial \boldsymbol{\xi}]$ is the Jacobian matrix mapping from the global coordinate system x_i to the intrinsic coordinate system ξ_j, and $\partial \xi_l/\partial x_k$ can be determined by the following matrix relationship [4]:

$$\left[\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{x}}\right] = [J]^{-1} = \left[\frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}}\right]^{-1} \tag{7}$$

where

$$\left[\frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}}\right]_{ij} = \frac{\partial x_i}{\partial \xi_j} = \frac{\partial N_\alpha}{\partial \xi_j}x_i^\alpha \tag{8}$$

The isoparametric elements used in FEM [4,5] have excellent features in geometry expression and physical variable interpolation. Referring to Ref. [25], the first and second partial derivatives needed in solving PDEs can be determined analytically by using the shape functions of isoparametric elements and a system of equations can be formed by substituting these spatial derivatives into the governing equation and boundary conditions.

4. Assembling system of equations from governing equations and boundary conditions

When solving a boundary value problem managed by a partial differential equation using EDM, the computational domain needs to be discretized into a series of isoparametric elements and nodes as done in FEM. The first and second spatial derivatives of physical variables can be calculated using Eqs. (4)-(6). Based on this, a system of equations can be directly formed by substituting Eq. (4) into the governing equation for internal nodes and boundary conditions for boundary nodes.

4.1. Setting up equations for internal nodes of elements based on the governing differential equation

For nodes located within an element, the governing equation (1) should be satisfied. To facilitate the use of EDM, Eq. (1) can be written as

$$\frac{\partial k_{ij}(\mathbf{x})}{\partial x_i}\frac{\partial T(\mathbf{x})}{\partial x_j} + k_{ij}(\mathbf{x})\frac{\partial^2 T(\mathbf{x})}{\partial x_i \partial x_j} + Q(\mathbf{x}) = \rho c\left(\frac{\partial T(\mathbf{x})}{\partial t}\right) \tag{9}$$

Substituting Eq. (4) into Eq. (9) leads to

$$\left[\frac{\partial k_{ij}(\xi)}{\partial x_i}\frac{\partial N_\alpha(\xi)}{\partial x_j} + k_{ij}(\xi)\frac{\partial^2 N_\alpha(\xi)}{\partial x_i \partial x_j}\right]T^\alpha + Q(\xi) = \rho c\left(\frac{\partial T}{\partial t}\right) \quad \xi \in \Omega \tag{10}$$

where ξ is the intrinsic coordinate at the node inside the element. The term $\partial k_{ij}/\partial x_i$ in Eq. (10) can be calculated through direct differentiation, if the analytical expression of the heat conductivity k_{ij} is given, otherwise the following expression can be used to compute its value.

$$\frac{\partial k_{ij}(\xi)}{\partial x_i} = \frac{\partial N_\beta(\xi)}{\partial x_i}k_{ij}^\beta \tag{11}$$