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Analytic solution of Guyer-Krumhansl equation for laser flash experiments

R. Kovács *

Department of Theoretical Physics, Wigner Research Centre for Physics, Institute for Particle and Nuclear Physics, Budapest, Hungary
Department of Energy Engineering, Faculty of Mechanical Engineering, BME, Budapest, Hungary
Montavid Thermodynamic Research Group, Hungary

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The existence of non-Fourier heat conduction is known for a long time in small and low temperature systems. The deviation from Fourier's law has been found at room temperature in heterogeneous materials like rocks and metal foams (Both et al., 2016; Ván et al., 2017). These experiments emphasized that the so-called Guyer-Krumhansl equation is adequate for modeling complex materials. In this paper an analytic solution of Guyer-Krumhansl equation is presented considering boundary conditions from laser flash experiment. The solutions are validated with the help of a numerical code (Kovács et al., 2015) developed for generalized heat equations.

1. Introduction

The existence of non-Fourier heat conduction under various conditions is experimentally proved in several different ways. First, the Cattaneo equation, also known as Maxwell–Cattaneo-Vernotte equation (MCV) [4–8]

$$\tau_q \partial_{tt} T + \partial_t T = \alpha \partial_{xx} T, \tag{1}$$

is used to describe the dissipative wave form of heat propagation called second sound. Here, τ_q is the relaxation time, α stands for the thermal diffusivity, ∂_t denotes the time derivative and ∂_{xx} denotes the second spatial derivative in one dimension. It is the simplest extension of Fourier's law and there are several different theorems in the literature which lead to this type of hyperbolic generalization [7–12,14,3,15–17]. The existence of second sound was predicted by Tisza and Landau [18,19], earlier than the experimental discovery. Then Peshkov managed to measure it in superfluid He [20] and enhanced the researches in that respect. Later on, several new ideas have developed how to measure similar phenomena in different materials. One of the most important result is related to Guyer and Krumhansl who derived the so-called window condition, significantly supporting the measurement of second sound in solids [21].

The next extension of Fourier's equation bears their names, called Guyer-Krumhansl (GK) equation [22–24],

$$\tau_q \partial_{tt} T + \partial_t T = \alpha \partial_{xx} T + \kappa^2 \partial_{txx} T, \tag{2}$$

where κ^2 is the dissipation parameter [3], strongly related to the mean free path from the aspect of kinetic theory [12]. It contains the MCV Eq. (1), however, it is a parabolic type model according to the classification of [13] and is able to recover the solution of Fourier equation when $\kappa^2/\tau = \alpha$ holds, called Fourier resonance [1,2,25]. Despite of the disadvantageous infinite propagation speed of parabolic models, it is still a valid and thermodynamically consistent realisation of non-Fourier heat conduction at room temperature [1,2,26].

Regarding the experiments, one should mention the ballistic-type heat conduction measured by Jackson et al. [27–30] in NaF crystals and modeled by several authors [31–34]. The most recent one can be found in [35] where quantitative agreement is obtained between the theory and experiments. The theory is based on non-equilibrium thermodynamics with internal variables and Nyíri multipliers [3,15,36].

The experimental success of measuring the second sound and the universal theory of non-equilibrium thermodynamics has motivated the researchers to find non-Fourier heat conduction in wave form described by the MCV Eq. (1) at room temperature. For example, such an endeavor is related to the experiments of Mitra et al. [37] where a frozen meat is used to find similar phenomenon. Unfortunately, no one was able to reproduce these experimental results and the measurements of Mitra et al. are

* Address: Department of Theoretical Physics, Wigner Research Centre for Physics, Institute for Particle and Nuclear Physics, Budapest, Hungary.
E-mail address: kovacs.robert@wigner.mta.hu

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widely criticized [38–40]. However, it turned out that the GK equation could be the relevant measurable extension of Fourier's law, the related non-Fourier effects are measured several times in different materials [1,2]. In many other cases the dual phase lag model is considered also as an adequate generalization [41–45], however, this model is contradictory to basic physical principles [26] and its validity is questionable [46–52].

All the aforementioned experiments are the heat pulse type, the underlying principle is the same, only the equipment is different. It is a standard method to measure the thermal diffusivity and is used widely in engineering practice. The importance of Guyer-Krumhansl Eq. (2) in the evaluation of such experiments indicated the need to find an analytic solution.

The work of Zhukovsky has to be mentioned here [53–56]. Recently, Zhukovsky obtained an exact solution of GK equation using operational method for infinite spatial domain. However, initial conditions are different from studied in this paper. Nevertheless, these results are still far from applicability for evaluating experiments. Therefore, the goal of this paper is to complement the results of the aforementioned papers to be more applicable for real experimental setup like described below.

2. Experimental setup and boundary conditions

Measurements finding non-Fourier heat conduction in heterogeneous materials are performed at room temperature as it is described in detail in the papers [1,2] have the following setup, see Fig. 1.

The front side boundary condition depicts the heat pulse which excites the heterogeneous sample. The pulse has a finite length, given as $t_p = 0.01$ s [1,2]. The exact shape of the pulse has not been taken in account in [1,2] during the evaluation process, nevertheless, its length is critical and greatly influences the solution [57,58]. As it is highlighted and applied in [1–3,59], the following function is considered to model the heat pulse,

$$q(x=0,t) = \begin{cases} q_{\max}\left(1-\cos\left(2\pi\cdot\frac{t}{t_p}\right)\right) & \text{if } 0 \leqslant t \leqslant t_p, \\ 0 & \text{if } t > t_p, \end{cases}$$

that is, the front side boundary condition is given by prescribing the heat flux in time, here q_{max} is the amplitude of the signal. When the experimental results are evaluated, the cooling on rear boundary has to be considered: $q(x=L,t) = h(T - T_\infty)$, where h is the heat transfer coefficient and T_∞ is the constant ambient temperature.

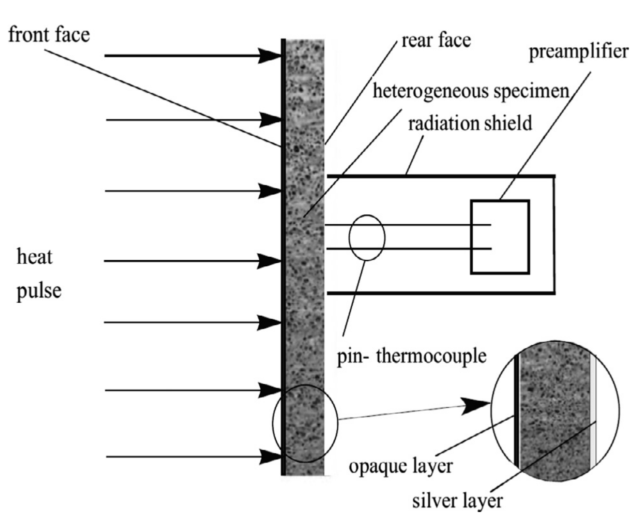

Fig. 1. Arrangement of the experiment, original figure from [2].

In spite of the fact that it is crucial to model these effects, in the analytic solution it is neglected to simplify the mathematical problem. Thereby adiabatic condition is applied to the rear side for every time instant: $q(x=L,t) = 0$. Regarding the initial conditions, all the time derivatives are zero at the initial state and the sample is in equilibrium with its environment, i.e.:

$$T(x,t=0) = T_0, \quad q(x,t=0) = 0, \quad \partial_t T(x,t=0) = 0,$$
$$\partial_t q(x,t=0) = 0. \tag{3}$$

3. Dimensionless quantities

In order to ease the solution of GK equation dimensionless quantities are used (see [3] for details). From now on, the same formalism is applied, that is, the following parameters are introduced,

$$\hat{t} = \frac{\alpha t}{L^2} \quad \text{with} \quad \alpha = \frac{\lambda}{\rho c}; \quad \hat{x} = \frac{x}{L};$$
$$\widehat{T} = \frac{T - T_0}{T_{\text{end}} - T_0} \quad \text{with} \quad T_{\text{end}} = T_0 + \frac{\bar{q}_0 t_p}{\rho c L};$$
$$\hat{q} = \frac{q}{\bar{q}_0} \quad \text{with} \quad \bar{q}_0 = \frac{1}{t_p}\int_0^{t_p} q_0(t)dt, \tag{4}$$

where L is the length of the sample, λ, ρ and c are the thermal conductivity, mass density and specific heat, respectively. The time averaged heat flux $\bar{q}_0$ is used to define the equilibrium temperature T_{end}. The material parameters converted with

$$\hat{\tau}_\Delta = \frac{\alpha t_p}{L^2}; \quad \hat{\tau}_q = \frac{\alpha \tau_q}{L^2}; \quad \hat{\kappa} = \frac{\kappa}{L}, \tag{5}$$

where $\hat{\tau}_\Delta$ stands for the dimensionless heat pulse length and $\hat{\tau}_q$ denotes the relaxation time related to the heat flux. For the sake of simplicity, the notation "hat" is omitted and let us restrict ourselves only for dimensionless quantities. Using these formalism the GK-type heat equation reads as

$$\tau_q \partial_{tt} T + \partial_t T = \partial_{xx} T + \kappa^2 \partial_{txx} T, \tag{6}$$

which can be decomposed into two equations containing the balance equation of internal energy

$$\tau_\Delta \partial_t T + \partial_x q = 0, \tag{7}$$

and the GK-type constitutive equation is:

$$\tau_q \partial_t q + q + \tau_\Delta \partial_x T - \kappa^2 \partial_{xx} q = 0. \tag{8}$$

Since the boundary conditions are prescribed as a given heat flux in time it is suitable to eliminate T from the Eqs. (7) and (8):

$$\tau_q \partial_{tt} q + \partial_t q = \partial_{xx} q + \kappa^2 \partial_{txx} q. \tag{9}$$

After obtaining the solution for $q(x,t)$ one can use Eq. (7) to integrate $\partial_x q$ with respect to time and calculate $T(x,t)$. Applying dimensionless quantities, the heat pulse boundary condition at the front side reads as

$$q(x=0,t) = q_0(t) = \begin{cases} \left(1-\cos\left(2\pi\cdot\frac{t}{\tau_\Delta}\right)\right) & \text{if } 0 \leqslant t \leqslant \tau_\Delta, \\ 0 & \text{if } t > \tau_\Delta, \end{cases}$$

and for the rear side $q(x=1,t) = q_L(t) = 0$ holds together with the dimensionless initial conditions $T(x,t=0) = 0$ and $q(x,t=0) = 0, \partial_t T(x,t=0) = 0, \partial_t q(x,t=0) = 0$.

4. Solution method

According to the front side boundary condition it is reasonable to split the solution into two sections in time. The first one goes