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A modified Lie-group shooting method for multi-dimensional backward heat conduction problems under long time span

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ABSTRACT

This paper proposes a modified Lie-group shooting method to solve multi-dimensional backward heat conduction problems under long time spans. The backward heat conduction problem is renowned for being ill posed because the solutions are generally unstable and highly dependent on the given data. For dealing with those problems, the Lie-group shooting method is one of the most powerful tools to find the unknown initial condition for the backward heat conduction problems in the time domain. In previous studies, the Lie-group shooting method uses the time and spatial semi-discretization technique to change the integration direction of numerical schemes and then increase the time span. However, the conversional Lie-group shooting method cannot get to the core of divergence problems for the backward heat conduction problems, especially the increased computational time. The main reason is that a real single-parameter Lie-group element occurs at zero, and a generalized midpoint Lie-group element is not equivalent to the single-parameter Lie-group element in the Minkowski space. Hence, to overcome the above problems, the relationship of the initial condition, the final condition and a real single-parameter r is assessed. According to the constraint condition of the initial and final condition, a real single-parameter r depends on the time span to maintain the numerical convergence. Again, in order to preserve the same Lie-group property in the time direction, the high-order Lie-group scheme based on the *generalized linear group* in Euclidean space is introduced, which concurrently satisfies the constraint of the cone structure, the Lie-group and the Lie algebra at each time step. The accuracy and efficiency are validated, even under noisy measurement data, by comparing the estimation results with existing literature.

1. Introduction

In many engineering application fields, it is important to find the temperature history or physical quantities from known measurement data. If the boundary, the initial conditions, the heat source-sink terms, and the physical properties of the material are specified, then this leads to a well-posed problem that may easily be dealt with by various numerical methods. This is referred to as the direct heat conduction problem (DHCP). However, in many practical situations, it is not always possible to specify the boundary condition, the initial temperature, the source-sink terms, and/or the material properties. This is referred to as the backward heat conduction problem (BHCP). Mathematically, BHCPs are classified as ill-posed problems because the solution is unstable for the given input data (see Payne [1]).

Numerical methods adopted for backward problems are usually implicit because the explicit schemes are apparently not very effective up to now. As mentioned by Ames and Epperson [2], ill condition from a numerical point of view are necessary for iterative methods, and the problem must be regularized before any approximation can be constructed. Obviously, an ill-posed problem is impossible to solve using classical numerical methods and requires special techniques to be employed. Several methods have been proposed in the literature to address these problems. For example, Han et al. [3] used the boundary element method (BEM) combined with a minimal energy technique to resolve the homogeneous BHCP. Lesnic et al. [4], Mera et al. [5,6], and Jourhmane and Mera [7] used the iterative BEM for homogeneous BHCPs. However, these approaches still cannot avoid the effect of ill condition when increasing discrete boundary nodes. Regularization approaches [8,9] have been widely proposed and applied, including the conjugate gradient method with an adjoint equation [10–12], the regularized solution using a quasi-Newton method, and the regularized solution using the genetic algorithm (GA) method. Muniz et al. [9] adopted Tikhonov regularization, the maximum entropy principle, and truncated singular value

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Nomenclature

A	a vector
b	a vector
$\mathbf{a}_m$	the coefficient defined in Eq. (21)
$\mathbf{b}_m$	the coefficient defined in Eq. (21)
$\mathbf{A}$	the coefficient matrix
$\mathbf{f}$	an n-dimensional vector field
$\mathbf{u}$	an n-dimensional vector field
$\mathbf{u_0}$	initial temperature vector
$\mathbf{u_f}$	temperature vector at final time t_f
$\hat{\mathbf{u}}$	$:= r\mathbf{u_0} + (1-r)\mathbf{u_f}$
$\hat{t}$	$:= (1-r)t_f$
R	the set of real numbers
R^n	an n-dimensional Euclidean space
Ω	a space-time domain
D	a bounded domain in R^n
$\otimes$	a dyadic operation
$gl(n, R)$	a real Lie algebra
$GL(n, R)$	the general linear group
$\mathbf{I}_n$	an n-dimensional unit matrix
$\mathbf{G}$	an element of a Lie group
u	the temperature distribution
t	a temporal coordinate
x	a spatial variable
y	a spatial variable
z	a spatial variable
t	time
Δx	the lattice spacing length of x
Δy	the lattice spacing length of y
Δz	the lattice spacing length of z
Δt	a time increment
τ	iteration number
$\hat{\mathbf{E}}_m$	the n-dimensional vector field defined in Eq. (26)
t_f	the final time
$R(i)$	random numbers
σ	the noise level

Greek symbols

ε	a given stopping criterion
Λ_m	the variable coefficient defined in Eq. (14)
Θ_m	the variable coefficient defined in Eq. (23)
θ	intersection angle of $\mathbf{u_f} - \mathbf{u_0}$ and $\mathbf{u_0}$
u_b	given boundary data
u_f	given final data
u_0	given initial data

Subscripts and superscripts

i	spatial grid numbers in the x direction
j	spatial grid numbers in the y direction
k	spatial grid numbers in the z direction
m	grid number in the time direction
f	final
0	initial
T	transpose

decomposition to solve homogeneous BHCPs and obtained promising results. Mera [13] developed the method of fundamental solutions (MFS) and combined the method with the standard Tikhonov regularization technique to address BHCPs. These approaches with the regularization technique can obtain an approximate solution, but they still cannot to overcome increasing the computational time span and discrete boundary nodes. Very recently, the singular boundary method (SBM), which is a strong-form boundary collocation method, has been widely applied to deal with inverse problems. For example, Wang et al. [14–16] applied the SBM in conjunction with several regularization techniques to address 2D thin-walled structures and the 3D Cauchy problems of steady heat conduction. Li et al. [17] derived the explicit error bounds of the SBM and illustrated the physical meaning of the origin intensity factor in the SBM and BEM. Further, Li and Chen [18,19] applied the modified SBM to address the multi-dimensional wave problems and active noise control. The SBM can avoid the ill condition problem of the conventional iterative schemes and obtain the stable solution using very few boundary nodes and small CPU time.

In recent years, there has been a substantial development in the area of the geometric integration of the non-linear ordinary differential equations (ODEs) evolving on the Lie groups, and more generally on the homogeneous space. Basically, there are two principal approaches to accomplish the task. The first one is to use the classical numerical scheme to approximate the solution at a given point and then to project this result on the manifold. The second approach consists of the construction of numerical methods whose solutions automatically evolve on a given manifold. Therefore, it is important to derive the numerical method whose solutions must evolve on the manifold. To preserve the solutions of the numerical method evolved on the manifold, Lie groups play a key role. By sharing the geometry and invariance with the ODEs, the numerical methods are more accurate and more effective than conventional ones. In order to retain the invariance of the underlying dynamical system in the Minkowski space, Liu [20] has developed numerical method, which is called the group preserving scheme (GPS), to integrate the augmented dynamical system of ordinary differential equations that evolve on a matrix Lie-group $SO_o(n, 1)$. Liu [21] proved that the implicit and explicit Lie-group schemes based on $GL(n, R)$ are equivalent to the GPS in Minkowski space. There are three types of properties in the GPS: cone construction, Lie algebra, and group properties. The past studies [22–25] clearly indicated that the Lie-group methods not only produce an improved qualitative behaviour but also allow for a more accurate long term integration than that offered by the general purpose methods.

Recently, Liu et al. [26,27] applied the backward GPS to address homogeneous BHCPs. Chang and Liu [28] applied the BGPS to deal with multi-dimensional backward wave problems. However, a BGPS cannot efficiently avoid to numerical errors propagation in integration direction of the numerical schemes. Chang et al. [29] and Chang et al. [30] proposed the Lie-group shooting method (LGSM) for the quasi-boundary regularization of multi-dimensional BHCPs. However, the LGSM with the regularization parameter cannot avoid numerical divergence when increasing the computational time and discrete element numbers. Liu [31] employed a spatial-direction LGSM to address the 1D BHCP, while Liu and Chang [32] used the $GL(n, R)$ scheme to recover an unknown initial temperature for a non-linear heat conduction equation. Although changing the integration direction of numerical schemes can increase the computational time, the numerical scheme is limited only for the 1D BHCPs. From the above numerical results, the solution obtained using the LGSM still suffers numerical divergence when the time space length increases. The main reason is that a real single-parameter Lie group element $G(t_f)$ occurs at zero, and a generalized midpoint Lie-group element $G(r) \neq G(t_f)$. Chen [33] applied the fictitious time integration method (FTIM) to solve the multi-dimensional BHCPs and assessed the relation of the initial and final condition to achieve one-step. To