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Discontinuous finite element method with a local numerical flux scheme for radiative transfer with strong inhomogeneity

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ABSTRACT

A discontinuous finite element method (DFEM) with a local numerical flux scheme is developed for solving radiative transfer problems in participating media with strongly inhomogeneous medium properties, steep gradient source, and inhomogeneous angular radiation intensities. The discrete elements in DFEM are assumed to be discontinuous on the inner-element boundaries and the shape functions are constructed on each element. The continuity of the computation domain is maintained by modeling a numerical flux across the inner-boundaries, which makes the DFEM suitable, accurate and numerical stable for radiative transfer problems involving strong inhomogeneity. Several test cases are studied to evaluate the DFEM performance for radiative transfer equation (RTE) with strong inhomogeneity. The DFEM solutions are compared with those obtained by the meshless method and the finite element method. Our results show that the DFEM is more accurate and stable than the other two methods.

1. Introduction

Radiative heat transfer in participating media has attracted significant attention due to its wide application in many scientific and engineering problems [1–3]. The radiative transfer equation (RTE) in an absorbing, emitting, and scattering medium is a complicated integral–differential equation which is difficult to find an analytical solution. Researchers, therefore, have been attempting numerical methods to obtain approximate solutions for the RTE. Several numerical techniques have been successfully applied to predict the radiative transfer process in participating media, such as the finite element method (FEM) [4–6], the Monte Carlo method (MCM) [7–10], the discrete ordinate method (DOM) [11–14], the finite volume method (FVM) [15–17], the distributions of ratios of energy scattered or reflected (DRESOR) method [18–20], and the lattice Boltzmann method (LBM) [21–23], to cite a few examples.

The RTE is a first-order integral–differential equation and is convection-dominated. The presence of the convection term in the RTE may induce strong instabilities to the numerical solutions. In many numerical methods based on the discretization of the RTE, including the DOM and the FEM, the solution oscillations will occur if no special stability treatment is applied. Luo et al. [24] applied an upwind scheme to the direct collocation method to mitigate the solution oscillations. Zhao et al. [25] derived a second-order radiative transfer equation (SORTE) and presented its stable solutions obtained by the FEM. To overcome the singularity problem of SORTE for dealing with media with zero extinction coefficients, Zhao et al. [26] developed a modified second order form of radiative transfer equation (MSORTE) and tested its numerical performance for several critical cases.

For radiative transfer problems in media with strong inhomogeneity (e.g. inhomogeneous medium properties and source terms), the radiative intensity has strong non-uniform or even discontinuous distributions. The solutions obtained by the conventional numerical methods suffer significantly from the instabilities [24–26]. More accurate and stable numerical algorithms are urged to be developed for this kind of radiative transfer problems. Currently, the grid-based FEM is a popular algorithm for solving the RTE in participating media because of its advantage of dealing with multidimensional problems flexibly and efficiently. However, the FEM is a global method where the inner-element boundaries are enforced to be continuous, which will induce additional errors and instabilities to the numerical solutions. Thus when dealing with radiative transfer problems involving strong intensity gradients or even discontinuities, a local method is expected such that the intensity property across the inner-element boundaries can be modeled more precisely.

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Originally proposed by Reed and Hill [27], the discontinuous finite element method (DFEM) combines the salient features of FEM and FVM [28] and has been successfully applied to solve various partial differential equations [29–35] as well as the RTE [36–39]. Compared to the global shape functions in the conventional FEM, the shape functions in the DFEM are constructed on each discrete element locally. Therefore, the RTE is solved for each element in the DFEM application such that element-wise solutions can be obtained. A numerical flux across the inner-boundaries is numerically implemented to connect the adjacent element, and this maintains the continuity of the computational domain The DFEM is thus a local method and has the advantages of ensuring geometric flexibilities and supporting locally adapted resolutions at the same time, which makes it suitable for solving radiative transfer problems involving space- and angular-dependent inhomogeneity.

In this paper, the DFEM with a local numerical flux scheme is applied to radiative transfer problems involving strong inhomogeneous radiation distributions. The discrete-ordinate form of RTE and its DFEM discretization are presented in Section 2. The accuracy of DFEM and its superiority to bate the solution oscillations are verified in Section 3 by comparing the DFEM solutions with those by other methods for several test cases. This work is concluded in Section 4.

2. Mathematical formulation

2.1. Governing equation of radiative transfer

The discrete-ordinate form of radiative transfer equation, which describes the energy balance in the discrete direction, can be written as [1,2]

$$\boldsymbol{\Omega}^m \cdot \nabla I(\mathbf{r}, \boldsymbol{\Omega}^m) + \beta(\mathbf{r}) I(\mathbf{r}, \boldsymbol{\Omega}^m) = S(\mathbf{r}, \boldsymbol{\Omega}^m), \quad (1)$$

where $\boldsymbol{\Omega}^m = \mu^m \mathbf{i} + \eta^m \mathbf{j} + \xi^m \mathbf{k}$ the unit direction vector with $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ denoting the unit vector in the dimension of x, y, and z, respectively, I is the radiative intensity to be solved, $\mathbf{r}$ is the location, β is the extinction coefficient, and S is the source term defined as

$$S(\mathbf{r}, \boldsymbol{\Omega}^m) = \kappa_a(\mathbf{r}) I_b(\mathbf{r}) + \frac{\kappa_s}{4\pi} \sum_{m'=1}^{M} I(\mathbf{r}, \boldsymbol{\Omega}^{m'}) \Phi(\boldsymbol{\Omega}^{m'}, \boldsymbol{\Omega}^m) w^{m'}, \quad (2)$$

where κ_a and κ_s are the absorption coefficient and scattering coefficient respectively, I_b is the black body emission, Φ is the scattering phase function, and $w^{m'}$ is the weight corresponding to the direction $\Omega^{m'}$.

Consider the boundary emission and reflection, the boundary conditions are given as

$$I(\mathbf{r}_w, \boldsymbol{\Omega}^m) = \varepsilon_w I_b(\mathbf{r}_w) + \frac{1-\varepsilon_w}{\pi} \sum_{\mathbf{n}_w \cdot \boldsymbol{\Omega}^{m'} > 0} I(\mathbf{r}_w, \boldsymbol{\Omega}^{m'}) |\mathbf{n}_w \cdot \boldsymbol{\Omega}^{m'}| w^{m'} + \rho_s I(\mathbf{r}_w, \boldsymbol{\Omega}^{m''}), \quad (3)$$

where the subscript 'w' denotes the physical boundary, $\mathbf{n}_w$ denotes the unit outward normal vector on the physical boundary, $\Omega^{m''}$ denotes the corresponding incident direction of the current specular reflected radiation in the direction $\boldsymbol{\Omega}^m$.

2.2. DFEM discretization of RTE

The application of a grid-based algorithm for RTE starts with dividing the continuous computational domain into a finite number of elements. The complex partial differential equation is broken down into a series of linear simultaneous equations on the discrete elements. The continuous problem, which has an infinite number of unknowns, is thus reduced to a problem with a finite number of solutions on specific nodes.

In the conventional FEM, as shown in Fig. 1(a), the inner-element boundaries are enforced to be continuous, that is, the radiation intensities on the node shared by two adjacent elements are assumed to be the same. However, this assumption does not represent the nature of the solutions for the simultaneous equation and may bring large errors to radiative transfer problems with steep intensity gradients. In the DFEM application, the discontinuity across the inner-element boundaries is allowed, which means that at the same geometric point, the radiation intensities are considered to be discontinuous, as shown in Fig. 1(b). The approximation functions are constructed on each element and the solutions of RTE are element-wise, which matches the properties of the simultaneous equations on the discrete elements and makes the DFEM suitable for radiative transfer problems with inhomogeneity.

To gain the generality of DFEM discretization of RTE for multi-dimensional problems, a two-dimensional medium divided into triangular meshes (see Fig. 2) is used to illustrate the DFEM discretization procedure. By using the weight function locally defined on each element, Eq. (1) is weighted by $W(\mathbf{r}, \boldsymbol{\Omega})$ and is integrated over the element e using the Gauss divergence theorem [40]

$$-\langle I^m, \boldsymbol{\Omega}^m \cdot \nabla W \rangle_e + ([\boldsymbol{\Omega} I]^m \cdot \mathbf{n}_{\partial e}, W)_{\partial e} + \langle \beta I^m, W \rangle_e = \langle S^m, W \rangle_e, \quad (4)$$

where $[\boldsymbol{\Omega} I]^m$ denotes the numerical flux to be modeled, $\mathbf{n}_{\partial e}$ denotes the unit outward normal vector, the subscript ∂e denotes the element boundary, and the operators are defined as

$$\langle f, g \rangle_e = \int_e fg\, dV, \quad (f, g)_{\partial e} = \int_{\partial e} fg\, dA. \quad (5)$$

In this paper, the local Lax-Friedrichs numerical scheme [41] is chosen as the numerical flux across the elements, and it is expressed as

$$[\boldsymbol{\Omega} I]^m = \boldsymbol{\Omega}^m \bar{I}^m + |\boldsymbol{\Omega}^m| I^m \mathbf{n}_{\partial K}, \quad (6)$$

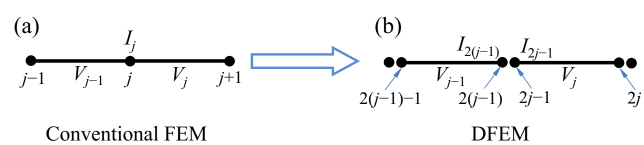

Fig. 1. The elements and solution nodes. (a) In the conventional FEM, the control volume of the jth element is $V_j = [j, j+1]$, the neighboring elements V_{j-1} and V_j share a solution node j where the radiation intensity keeps the same for the two elements, (b) in the DFEM, the control volume of the jth element is $V_j = [2j-1, 2j]$, each element has its own solution nodes. For the nodes $2(j-1)$ and $2j-1$, the geometric coordinates are the same while the radiation intensities on them can be different.

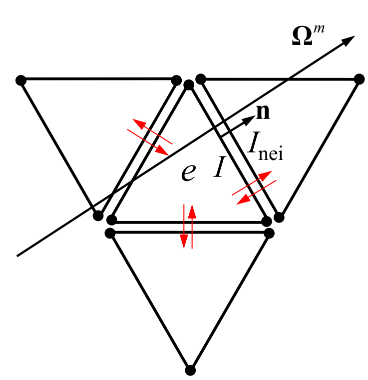

Fig. 2. Sketch of triangular meshes, unit outward normal vector of the element boundary, and the radiation intensity.