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The thermal and laminar boundary layer flow over prolate and oblate spheroids

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ABSTRACT

This paper is concerned with the thermal and laminar boundary layer flow over rotating axisymmetric bodies in an otherwise still fluid. The thermal boundary layer flow equations for two types of spheroids (prolate and oblate) are formulated. Similarly to the previous work of other researchers, the laminar flow equations for two types of spheroids are also reproduced. Furthermore, the series solutions are derived for the thermal and laminar flow equations for each body.

The series solutions are numerically calculated and the laminar flow profiles for each spheroid along with its heat transfer profile are visualized in detail. It is shown that for increasing eccentricity, the convection effects are increasing at higher latitudes of prolate spheroids, while the conduction effects are dominated at these higher latitudes of oblate spheroids. The results for spheres are reproduced in the case of zero eccentricity.

By using commercial routines from NAG, the solutions for rotating spheres in the case of laminar flow are also mentioned as in the previous existing works. In particular, the results from the transformation technique of solutions are in good agreement with those obtained from NAG in the case of spheres, particularly at lower latitudes for all eccentricities.

1. Introduction

Three-dimensional (3-D) boundary layer flow transition over rotating disks has been a subject of numerous studies; see, for instance, [1–9]. These studies are served as the foremost model problem for the subsequent investigations of the 3-D boundary layer flows over axisymmetric bodies of revolution. Both theoretically and experimentally, the case of a flow field structure of the laminar boundary layer flow over rotating spheres has been greatly clarified in the investigations of [10–16]. The flow visualization studies led by the papers [17–23] are related to the transition of the laminar boundary layer flow over rotating spheres and cones. The stability and instability of the boundary layer flows on rotating spheres, spheroids, and disks were reported by Garrett [24], Samad and Garrett [25], and Griffiths et al. [26].

The theoretical studies of [2,3,27–34] related to the transition phenomena of the laminar boundary layer flow over various rotating geometries like disk, sphere and cone were carried out in such a way that the governing laminar flow equations were first derived using some appropriate coordinate systems for each geometry. These laminar flow equations are actually a set of simultaneous 3-D nonlinear partial differential equations and are solved by using advanced numerical methods. Subsequently, the perturbation equations that govern the transition of the laminar boundary layer are derived for each body. The solutions of the laminar flow equations are then used in solving the related perturbation equations for each body. Recently, Samad [35] and Samad and Garrett [36,37] used the techniques in the aforementioned investigations and successfully derived the laminar flow equations for the general families of rotating prolate spheroids and oblate spheroids. The solutions were then used in the transition analysis of the laminar boundary layer flow over the general families of each type of spheroids.

The heat transfer problem from a rotating disk maintained at a constant temperature was first considered by Millsaps and Pohlhausen [38], for different Prandtl numbers in the steady state. The steady state heat transfer from a rotating disk maintained at

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a constant temperature to fluids at a variety of Prandtl numbers was also discussed by Sparrow and Gregg [39]. The thermal laminar boundary layer on a rotating sphere was investigated numerically in the work of Banks [40] which we rely and can be reproduced without any mathematical flaw. In this work, the results were matched with the experimental work of Kreith et al. [41] for rotating spheres. However, Banks [40] also investigated the case when surface temperature is non-uniform for a rotating sphere which we do not consider for spheroids. We only extend the results reported by Banks [40] for the uniformly heated spheres to all types of spheroids. In this paper, we calculate the basic flow profiles along with the heat transfer profile for both types of spheroids (prolate and oblate) using a simple transformation and we match our results in the case of spheres with those of Banks [40] and Kreith et al. [41].

2. Laminar and thermal boundary layer over spheroids

In this section, we show the laminar boundary layer flow equations for both types of regular spheroids, i.e., prolate and oblate spheroids from Samad [35] and Samad and Garrett [37] for consistency matter. Indeed, we extend the work related to the thermal boundary layer over spheres by Banks [40] to the general geometry of rotating spheroids (prolate and oblate) in an otherwise still fluid. The thermal boundary layer equations for both types of spheroids are formulated for the first time in this work. The series solutions are developed for the heat equations. However, these equations are solved simultaneously with the series solutions of the laminar flow equations for prolate and oblate spheroids from the work of Samad and Garrett [37]. The new transformations are also introduced in this paper to reduce partial differential equations that govern the thermal and laminar flow equations for each prolate spheroid and oblate spheroid to a set of ordinary differential equations. Furthermore, these ordinary differential equations will be solved numerically.

We show the governing laminar flow equations for prolate and oblate spheroids from the previous related works of others, and the derivation of the thermal boundary layer equation has been shown in detail. Furthermore, the new transformations to reduce the governing partial differential equations of the laminar boundary layer flow over prolate and oblate spheroids to a system of ordinary differential equations are shown.

2.1. The formulation for prolate and oblate spheroids

In 2010, Samad and Garrett [37] introduced two different orthogonal curvilinear coordinate systems, which are called prolate and oblate spheroidal coordinate systems. They converted the full Navier–Stokes equations into these coordinate systems. By applying the Prandtl boundary layer assumptions the governing laminar flow equations of the incompressible boundary layer over rotating prolate and oblate spheroids were developed in an otherwise still fluid in the steady state. We derive the governing thermal boundary layer equations for both types of spheroids by extending the technique used for rotating spheres (see Banks [40]).

For prolate spheroids, a prolate spheroidal coordinate system has been used and is defined relatively to the cartesian coordinate system as

$$x^\star = \sqrt{\eta^{\star 2} - d^{\star 2}}\sin\theta\cos\phi,$$
$$y^\star = \sqrt{\eta^{\star 2} - d^{\star 2}}\sin\theta\sin\phi,$$
$$z^\star = \eta^\star\cos\theta.$$

Similarly, for oblate spheroids, an oblate spheroidal coordinate system has been used and is related to the cartesian coordinate system as

$$x^\star = \eta^\star\sin\theta\cos\phi,$$
$$y^\star = \eta^\star\sin\theta\sin\phi,$$
$$z^\star = \sqrt{\eta^{\star 2} - d^{\star 2}}\cos\theta.$$

In both coordinate systems, $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi \leqslant 2\pi$. Note that $d^\star$ is the focal distance of each of the cross-sectional ellipses over each spheroid and $\eta^\star$ is the length of the normal at the surface (at point (θ,ϕ)). Note that $\eta_0^\star$ is the length of the normal at a particular latitude θ and azimuth ϕ of a particular body (prolate and oblate). For further explanation, we refer the reader to the paper by Samad and Garrett [37]. The prolate and oblate spheroids rotate in still fluid with a constant angular velocity $\Omega^\star$.

The eccentricity of the cross-sectional ellipse of each spheroid is denoted by e and is defined as $e = d^\star/\eta^\star$. Samad and Garrett [37] converted the full Navier–Stokes equations into both of the above mentioned orthogonal coordinate systems and derived the dimensional laminar flow equations for each body. Let $U^\star, V^\star$ and $W^\star$ be the velocities in θ-, ϕ- and $\eta^\star$-directions, respectively. Note that asterisks denote dimensional quantities. The dimensional laminar flow equations for both types of spheroids can be seen in the work of Samad [35]. However, we show the derivation of the dimensional thermal boundary layer over both types of spheroids in detail. The general energy equation for incompressible fluid with constant thermal conductivity k can be written as

$$\rho C_p \vec{V}\cdot\nabla T^\star = k\nabla^2 T^\star + \Phi, \tag{2.1}$$

where $T^\star$ is the local temperature and C_p is the specific heat. Furthermore, Φ is the dissipation function and is ignored as we do not consider heating due to viscous effects. We write Eq. (2.1) into general orthogonal curvilinear coordinates (as Eq. (A.1) in Appendix A). This equation is then transformed separately into prolate spheroidal and oblate spheroidal coordinate systems. The full energy equations are transformed into both types of coordinate systems (see Eqs. (A.2) and (A.3) in Appendix A).

Similarly to the work of Samad and Garrett [37], we apply the Prandtl boundary layer assumptions that $U^\star \sim O(1)$, $V^\star \sim O(1)$, $W^\star \sim O(\delta^\star)$, $\partial/\partial\theta \sim O(1)$, $\partial/\partial\eta \sim \delta^{\star -1}$. Also, we assume that $T^\star \sim O(1)$ and $\delta^\star \sim O(\nu^\star/\Omega^\star)^{1/2}$. These boundary layer assumptions reduce the heat equation to dimensional equations that govern the thermal boundary layer over rotating prolate and oblate spheroids. These governing dimensional thermal boundary layer equations for both types of spheroids are shown as below. For prolate spheroids,

$$\rho C_p\left(\frac{U^\star}{\sqrt{\eta_0^{\star 2} - d_0^{\star 2}\cos^2\theta}}\frac{\partial T^\star}{\partial\theta} + \frac{W^\star\sqrt{\eta_0^{\star 2} - d_0^{\star 2}}}{\sqrt{\eta_0^{\star 2} - d_0^{\star 2}\cos^2\theta}}\frac{\partial T^\star}{\partial\eta^\star}\right) = k\frac{\eta_0^{\star 2} - d_0^{\star 2}}{\eta_0^{\star 2} - d_0^{\star 2}\cos^2\theta}\frac{\partial^2 T^\star}{\partial\eta^{\star 2}} \tag{2.2}$$

and, for oblate spheroids,

$$\rho C_p\left(\frac{U^\star}{\sqrt{\eta_0^{\star 2} - d_0^{\star 2}\sin^2\theta}}\frac{\partial T^\star}{\partial\theta} + \frac{W^\star\sqrt{\eta_0^{\star 2} - d_0^{\star 2}}}{\sqrt{\eta_0^{\star 2} - d_0^{\star 2}\sin^2\theta}}\frac{\partial T^\star}{\partial\eta^\star}\right) = k\frac{\eta_0^{\star 2} - d_0^{\star 2}}{\eta_0^{\star 2} - d_0^{\star 2}\sin^2\theta}\frac{\partial^2 T^\star}{\partial\eta^{\star 2}} \tag{2.3}$$