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A highly accurate backward-forward algorithm for multi-dimensional backward heat conduction problems in fictitious time domains

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ABSTRACT

This paper proposes highly accurate one-step backward-forward algorithms for solving multi-dimensional backward heat conduction problems (BHCPs). The BHCP is renowned for being ill-posed because the solutions are generally unstable and highly dependent on the given data. In this paper, the present algorithm combines algebraic equations with a high-order Lie-group scheme to construct one-step algorithms called the backward fictitious integrate method (BFTIM) and the forward fictitious integrate method (FFTIM). First, the original parabolic equation is transformed into a new parabolic equation of an evolution type by introducing a fictitious time variable. Then, the numerical integration of the discretized algebraic equations must satisfy the constraints of the cone structure, Lie-group and Lie algebra at each fictitious time step. Finally, the algorithms with the minimum fictitious time steps along the manifold of the Lie-group scheme approach the true solution with one step when given an initial guess. In addition, this paper provides a strategy to determine the initial guess, which is the reciprocal relationship of the initial condition (IC) and the final condition (FC). More importantly, the IC and FC can be recovered by the BFTIM and FFTIM according to the relation between the IC and FC, even under large noisy measurement data. Five numerical examples of the BHCP are tested and numerical results demonstrate that the present schemes are more effective and stable. In general, the numerical implementations of the BFTIM and FFTIM are simple and have one-step convergence speeds.

1. Introduction

Heat conduction problems (HCPs) in engineering applications are widely classified as direct heat conduction problems (DHCPs) and inverse heat conduction problems (IHCPs). For these problems, it is very difficult to obtain analytical and exact solutions. Therefore, highly accurate and efficient numerical methods for DHCPs have recently been developed, especially with finite element methods [1–3], finite volume methods [4–6], boundary element methods [7–9], and meshless methods [10–13]. Compared with mesh-dependent or meshless approaches, these approaches use different discrete techniques to increase the accuracy and stability of the numerical solution, although they cannot avoid numerical error accumulation and propagation in the time direction, especially with initial values containing noise effects.

An IHCP involves the estimation of physical quantities, such as boundary or initial conditions, source-sink terms, and material properties. These problems are referred to as backward heat conduction problems (BHCPs). Mathematically, BHCPs are classified as the most strongly ill-posed problems because their solutions are unstable for the given input data. Many researchers have studied BHCPs. Han et al. [14] used the boundary element method combined with a minimal energy technique to resolve the homogeneous BHCP. Lesnic et al. [15], Mera et al. [16,17], and Jourhmane and Mera [18] used the iterative boundary element method for homogeneous BHCPs. Muniz et al. [19] proposed an explicit inversion method and a sequential scheme of inversion to solve homogeneous BHCPs. Several investigators have solved BHCPs using various approaches discussed in the literature. However, unsolved numerical stability and multi-dimensional problems remain. Regularization approaches [19,20] have been widely proposed and applied, including the conjugate gradient method with an adjoint equation [21–23], the regularized solution using a quasi-Newton method, and the regularized solution using the genetic algorithm (GA) method. Muniz et al. [20] adopted Tikhonov regularization, the maximum entropy principle, and truncated singular value decomposition to solve homogeneous BHCPs and obtained promising results. Mera [24] developed the method of fundamental solutions (MFS) and combined the method with the standard Tikhonov regularization technique to address BHCPs. Liu [25] proposed an implicit method and the explicit difference scheme to solve forward and backward heat conduction

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Nomenclature

$\mathbf{a}$	a vector
$\mathbf{b}$	a vector
$\mathbf{a}_m$	the coefficient defined in Eq. (30)
$\mathbf{b}_m$	the coefficient defined in Eq. (30)
$\mathbf{A}$	the coefficient matrix
$\mathbf{L}$	an n-dimensional vector field
$\mathbf{U}$	an n-dimensional vector field
R	the set of real numbers
R^n	an n dimensional Euclidean space
Ω	a space–time domain
D	a bounded domain in R^n
$gl(n,\mathrm{R})$	a real Lie algebra
$GL(n,\mathrm{R})$	the general linear group
$H(u)$	a nonlinear function
$\mathbf{I}_n$	an n-dimensional unit matrix
$\mathbf{G}$	an element of a Lie group
u	the temperature distribution
S	a heat source
t	a temporal coordinate
x	a spatial variable
y	a spatial variable
z	a spatial variable
t	time
Δx	the lattice spacing length of x
Δy	the lattice spacing length of y
Δz	the lattice spacing length of z
Δt	a time increment
$\Delta \tau$	a fictitious time increment
τ_m	$m\Delta\tau$
$\hat{\mathbf{E}}_m$	the n-dimensional vector field defined in Eq. (35)
t_f	the final time
$R(i)$	random numbers
σ	the noise level
m_1	the number of subintervals in the time direction
m_2	the number of grid points in each spatial direction

Greek symbols

ε	a given stopping criterion
Λ_k	the variable coefficient defined in Eq. (23)
Θ_k	the variable coefficient defined in Eq. (28)
Ψ	weighting factor
α	a thermal conductivity coefficient
ν	a viscosity-damping coefficient
u_b	given boundary data
u_f	given final data
u_0	given initial data

Subscripts and superscripts

i	spatial grid numbers in the x direction
j	spatial grid numbers in the y direction
k	spatial grid numbers in the z direction
ℓ	grid number in the time direction
m	index
b	boundary condition
f	final
0	initial
T	transpose

equations. However, the iterative methods, the meshless method, and the Tikhonov regularization technique with the L-curve method still exhibit numerical stability problems. Very recently, Wang et al. [26–28] applied the singular boundary method (SBM) in conjunction with several regularization techniques to deal with 2D thin-walled structures and 3D Cauchy problems of steady heat conduction. The strategy of SBM can avoid the numerical instability, and the precision does not change with the computational domain. For long time spans and final conditions with large noise levels, the BHCPs have been very difficult to solve.

Recently, a numerical method with a special structure was applied to handle DHCPs and IHCPs very effectively. Liu et al. [29,30] applied the backward group-preserving scheme (BGPS) to address homogeneous BHCPs. Chang et al. [31] proposed the Lie-group shooting method (LGSM) for the quasi-boundary regularization of multi-dimensional BHCPs. Liu [32] employed a spatial-direction LGSM to address 1-D BHCPs; Liu and Chang [33] used the $GL(n,\mathrm{R})$ scheme to recover an unknown initial temperature for a 1-D nonlinear BHCP. Although the $GL(n,\mathrm{R})$ scheme can handle long time spans and initial conditions with noise disturbances, it cannot address the multi-dimensional BHCPs. From the above numerical results, the solutions obtained using the LGSM still suffer from noise propagation, and the time step increment, the convergent criterion, and the lattice spacing length are important parameters that must be chosen. Thus, these numerical schemes with special structures cannot avoid integration paths in space and time.

To overcome these problems, Liu and Atluri [34] developed a fictitious time integration method (FTIM) to solve large systems of nonlinear algebraic equations and showed that this method could achieve high performance. Chang [35] and Liu and Chang [36] further applied a fictitious time integration method for multi-dimensional backward heat conduction problems in Minkowski space. Although the approach provided good results, even for a large noise effect, it is difficult to choose the relevant parameters, such as the viscosity-damping coefficient, the fictitious time step, and the fictitious terminal time, especially the initial guess values. When these parameters are determined, numerical instability occurs that varies with time integration. Hence, this paper employs the FTIM combined with high-order explicit Lie-group schemes based on $GL(n,\mathrm{R})$ to preserve the space–time manifold and avoid determining any parameters, including the viscous damping coefficient, the fictitious time step, the fictitious terminal time, the convergent criterion, and the initial guess value.

To make the integration path of the algebraic equation along with the manifold of the group-preserving scheme (GPS), which was proposed by Liu [37], the present strategy develops the high-order GPS to integrate in time and over a fictitious time direction. Liu [38] proved that the implicit and explicit Lie-group schemes based on $GL(n,\mathrm{R})$ are equivalent to the GPS in Minkowski space. There are three types of properties in the GPS: cone construction, Lie algebra, and group properties. According to the manifold, the integration path of the FTIM must satisfy the GPS cone condition, which means that the FTIM must be kept on the surface of the cone such that the gradient is constrained according to the fictitious time integration path of the cone. When using the minimum fictitious time step, the high-order Lie-group scheme can preserve integration continuity and obtain the most computationally efficient results.

The remainder of this paper is organized as follows. Section 2 illustrates the mathematical formulation of the FTIM and constructs the high-order explicit $GL(n,\mathrm{R})$ Lie-group schemes. Five