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Exact harmonic solution to ballistic type heat propagation in thin films and wires

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ABSTRACT

The system of hyperbolic type differential equations is considered for non–Fourier heat transfer in thin films and wires. Structures of the harmonic solutions for telegrapher equation and for Guyer–Krumhansl equation are identified and compared with each other. The exact harmonic solution is obtained; the influence of the phonon heat transfer is explored. The maximum principle violation is demonstrated. Frequency dependence of the solution for Guyer–Krumhansl equation is shown. The exact analytical solution is derived for the inhomogeneous ballistic–diffusion set of differential equations, governing heat propagation in thin films. The contribution of the ballistic, wave and diffusive heat transfer components as well as the influence of the initial conditions and of the Knudsen number on heat conduction in thin films and wires are investigated. The cases of identical and different from each other Knudsen numbers for ballistic and diffusive components are explored. The examples of exact solutions for the heat transfer in each case are demonstrated.

1. Introduction

Notwithstanding the rapid progress in computer methods and common nowadays computer modeling of physical processes, this is all the more necessary that machine based approach is complemented by proved analytical solutions where possible. Analytical solutions for a physical problem depend of the mathematical formulation of the latter. Perhaps, the most common physical phenomena met around is heat conduction. In most cases it is basically described by the Fourier law $\partial_t T = \alpha \partial_x^2 T$ [1], which agrees with experiment and observation in solid non-deformable bodies at room temperature. However, deviations from Fourier law in some crystal dielectrics occur at low temperatures <25 K, and Fourier law does not hold in low dimensional systems, such as thin films and wires [2], and even in some highly inhomogeneous materials [3] at normal conditions. The important limitation of Fourier law consists in the total lack of inertia, so that the temperature change is instantly perceived everywhere in the body once it occurs in one distant point. Already Maxwell indicated a second order equation for heat conduction and Onsager [4] concluded that Fourier law was no more than an approximate description of heat conduction, which neglected the time, needed for the heat flow acceleration. One of the most remarkable phenomena, which stands out by its non-Fourier behavior, is the second sound [5], first discovered in liquid Helium and then confirmed in solid crystals [6–9]. It was described theoretically by phonon heat transfer. Cattaneo proposed the proper differential equation (DE), which reads in terms of temperature as follows [10]:

$$(\tau \partial_t^2 + \partial_t) T = k_T \nabla^2 T, \tag{1}$$

where k_T is the heat conductivity, τ is the relaxation time. Cattaneo's model supposes that heat propagates in media like damped waves at finite speed $v = \sqrt{k_T/\tau}$, where τ, being intrinsic property of the media, specifies the delay between moments of the temperature change and the heat flux reaction on it. This time represents in fact the unit of heat inertia of the media and it can be viewed as the characteristic time of phonon–phonon interactions. At normal conditions it is very small, $\tau \approx 10^{-13}$ s. The speed of the second sound v specifies the heat wave propagation speed in matter exactly as the sound wave does, while k_T defines the heat diffusion. Hyperbolic heat equation (HHE) with the constant term

$$\left(\frac{\partial^2}{\partial t^2} + \varepsilon \frac{\partial}{\partial t}\right) F(x,t) = \left(\alpha \frac{\partial^2}{\partial x^2} + \kappa\right) F(x,t), \ \varepsilon, \ \alpha, \ \kappa = \text{const}, \tag{2}$$

is otherwise called the telegrapher's equation as it describes the electric signal propagation in long electric lines without emission [11]. For heat conduction the following notations are common: $\tau = 1/\varepsilon$, $k_T = \alpha/\varepsilon$, $\mu = \kappa/\varepsilon$; in Cattaneo Eq. (1) $\mu = 0$. We will

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though keep $\kappa \neq 0$ for generality. It is profitable for applications as we will see in what follows.

Cattaneo equation as well as telegrapher's equation were solved both analytically and numerically. Cattaneo equation in infinite media with arbitrary boundary conditions was studied in [12]. Analytical solution of telegrapher's equation with periodic boundary conditions was found in [13,14]. Point–like heating of semi-infinite one-dimensional string was analyzed in [15]; heat propagation after symmetric heating of thin film was studied in [16]; two-dimensional problem for Cattaneo heat propagation in cylinder was solved in [17]. Usually initial conditions are imposed on the function and on its derivative: $F(x,0)$ and $\partial_t F(x,0)$ are given. Separation of variables is possible for hyperbolic heat equation [18,19]. The obtained solution appears in the form of series of periodic functions. Although HHE qualitatively describes second sound, it disagrees quantitatively with numerous experimental measurements. Further generalizations of the HHE emerged; one of the most common is the Guyer–Krumhansl heat equation [20,21].

In our preceding publications [22,23,24] we derived integral forms for some particular analytical solutions for HHE and for GK type equations, and we considered special initial conditions. In this paper we will obtain a more general harmonic solution for the above types of equations and with its help we will investigate heat conduction in such physically valuable system as thin film. Previously it has been shown that heat transport in a thin film is somewhat more complicated than just Cattaneo or pure Guyer–Krumhansl heat conduction. Chen developed in [25,26] a model, which involves free ballistic propagation of phonons. Lebon et al. [27] derived partial differential equations that can substitute more complicated Chen theory for heat transport in thin films and wires. The new model of Lebon et al. actually arrives to inhomogeneous system of partial differential equations (PDE), involving Guyer-Krumhansl equation with a particular source term for the propagation of ballistic component; it was numerically solved in [27]. Complementing these studies, we will address in what follows heat transport in a thin film and solve proper inhomogeneous system of hyperbolic Guyer–Krumhansl type equations analytically, which has not been done as yet. Moreover, we will explore the analytical solution for physically valid range of Knudsen number, which plays the key role in low dimensional systems, such as ultra thin films, wires. We explore the structure of the harmonic solution $f(x) \propto e^{inx}$, which is important not only for radio–electric related applications, but also for any function, expandable in Fourier series. We compare such solution for telegrapher's equation and for Guyer–Krumhansl type equation, needed to solve the DE system for thin films, and we demonstrate frequency dependence of the heat conduction. We show the maximum principle non-compliance for non–Fourier solutions, and the influence of the phonon heat transfer, as well as the effect of the initial conditions and of the Knudsen number on heat conduction in thin films.

2. Evolution of the harmonic solution for hyperbolic heat equation

By means of exponential differential operators the exact solutions were obtained for a variety of differential equations, including extended forms of Fourier equation [22,28–30] and other heat conduction equations as well as other second order DE [23,30–33]. For the second order DE of the following type:

$$\left(\frac{\partial^2}{\partial t^2} + \varepsilon(x)\frac{\partial}{\partial t}\right)F(x,t) = \widehat{D}(x)F(x,t), \tag{3}$$

where $\widehat{D}(x)$ is the differential operator, acting on the coordinate x, we obtain with the help of Laplace transforms the following particular solution:

$$F(x,t) \propto Ce^{-\frac{t}{2}\hat{\varepsilon}(x)}\frac{t}{4\sqrt{\pi}}\int_0^\infty \frac{d\xi}{\xi\sqrt{\xi}}e^{-\frac{t^2}{16\xi}}e^{-\xi\hat{\varepsilon}^2(x)}e^{-4\xi\widehat{D}(x)}f(x), \tag{4}$$

provided the integral converges. The integral involves the action of the exponential differential operators on the initial condition $F(x,0) = f(x)$, where C is a constant, determined from the other initial condition. For the operator $\widehat{D}(x) = \alpha\partial_x^2 + \kappa$ the particular solution for the telegrapher's equation (2) emerges. The particular solution (4) for the DE (2), vanishing at $t \to \infty$ on semi-infinite axis, was studied for a number of initial functions $f(x)$ in [23,24]. In the more general case we must consider the following solution:

$$F(x,t) = e^{-\frac{t\varepsilon}{2}}\left(e^{-\frac{t}{2}\sqrt{\varepsilon^2+4\widehat{D}(x)}}C_1(x) + e^{\frac{t}{2}\sqrt{\varepsilon^2+4\widehat{D}(x)}}C_2(x)\right), \tag{5}$$

where $\widehat{D}(x) = \alpha\partial_x^2 + \kappa$ and $C_{1,2}(x)$ are determined by the initial or boundary conditions. In some cases the second branch of the solution can be obtained by substituting simultaneously $t \to -t,\ \varepsilon \to -\varepsilon,\ \delta \to -\delta$ in (4) (see [34]). In what follows we consider the example of the harmonic solution $\propto e^{inx}$ (5) for Eq. (2), which can be used to construct the exact solution for any function, expandable in Fourier series or integrals. In particular, it is possible to apply it for the initial Dirac δ-function or Gaussian in order to describe analytically typical for heat conductivity measurements heat pulse propagation [35]. This application, however, remains beyond the scope of the present paper and will be addressed in dedicated forthcoming publications. The action of the operator $e^{t\partial_x^2}$ was studied in [36–38]; The exponential differential operator $e^{\widehat{D}(x)}$ does not add new harmonics to those, existing at the moment $t = 0$. In the harmonic ansatz $\propto e^{inx}Y(t)$ PDE (2) reduces to simple ODE:

$$Y''(t) + \varepsilon Y'(t) = (-\alpha n^2 + \kappa)Y(t). \tag{6}$$

Consider, for example, the following Cauchy conditions:

$$F(x,t)|_{t=0} = Ge^{inx}, \quad \partial F(x,t)/\partial t|_{t=0} = Be^{inx}. \tag{7}$$

Following [28], we obtain the exact harmonic solution for the telegrapher's Eq. (2)

$$F(x,t)|_{F(x)\propto e^{inx}} = B_1 e^{inx-\frac{t}{2}(\varepsilon+\sqrt{V})} + B_2 e^{inx-\frac{t}{2}(\varepsilon-\sqrt{V})}, \quad V = \varepsilon^2 + 4(\kappa - \alpha n^2), \tag{8}$$

where we relate the coefficients $B_1,\ B_2$ to the initial conditions at $t = 0$: $B_1 + B_2 = G$ and $B_1(\varepsilon + \sqrt{V}) + B_2(\varepsilon - \sqrt{V}) = -2B$. This yields the following explicit expressions for them:

$$B_1 = \frac{-2B + G(-\varepsilon + \sqrt{V})}{2\sqrt{V}}, \quad B_2 = \frac{2B + G(\varepsilon + \sqrt{V})}{2\sqrt{V}}. \tag{9}$$

For the telegrapher's equation applied to the electric circuit, its solution has the meaning of an electric current evolution; in the context of heat conduction the initial conditions must be real and the solution describes the space–time heat distribution. The validity of the solution can be easily verified by its substitution in Eq. (2) with account for (7). Dependently on the values of the parameters α, ε, κ, and on the initial conditions, the value of the quantity V, which depends on the harmonic number n, can be negative and the coefficients $B_1,\ B_2$ can be complex. Some examples of solutions to the telegrapher's equation with different from each other initial conditions are given in Figs. 1–3.

In the considered example of the telegrapher's equation with α = 7, κ = –0.5, ε = 11, n = 1, B = 1, with initial conditions $F|_{t=0} = e^{inx}$, $\partial F/\partial t|_{t=0} = \pm 5e^{inx}$ we have V = 91 and for G = 5, $F = -0.6007e^{-10.2697t+ix} + 1.6007e^{-0.730304t+ix}$ (see. Fig. 1 left), while for G = –5 in Fig. 1 on the right, $F = 0.447586e^{-10.2697t+ix} + 0.552414e^{-0.730304t+ix}$. If the initial derivative is positive, as shown in Fig. 1 left, the maximum principle, which states that the solution