International Journal of Heat and Mass Transfer 119 (2018) 542–550

Contents lists available at ScienceDirect

International Journal of Heat and Mass Transfer

journal homepage: www.elsevier.com/locate/ijhmt

Similarity type of general solution for one-dimensional heat conduction in the cylindrical coordinate

Yang Zhou [a,c,*], Xiao-xue Hu [a], Ting Li [a,c], Dong-hai Zhang [b], Guo-qing Zhou [a]

[a] State Key Laboratory for Geomechanics and Deep Underground Engineering, China University of Mining & Technology, Xuzhou, Jiangsu 221116, China
[b] School of Mechanics & Civil Engineering, China University of Mining & Technology, Xuzhou, Jiangsu 221116, China
[c] JiangSu Collaborative Innovation Center for Building Energy Saving and Construct Technology, Xuzhou, Jiangsu 221116, China

ARTICLE INFO

Article history:
Received 19 June 2017
Received in revised form 19 September 2017
Accepted 27 November 2017

Keywords:
Radial heat conduction
General solution
Similarity transformation
Kummer functions
Line source problem
Stefan problem

ABSTRACT

One-dimensional heat conduction process in the cylindrical coordinate is investigated, and a similarity type of general solution is developed using the Kummer functions. The limiting behaviors of the general solution are studied using the properties of the Kummer functions, and some useful identities are deduced. As applications of the general solution, an infinite line source problem under power-type initial and boundary conditions and a one-phase Stefan problem with space-dependent latent heat in the cylindrical coordinate are studied. The analytical solutions for these two problems are established using the general solution directly. Computational examples for the analytical solutions are presented. For the infinite line source problem, the computational results are compared with those of the solid cylindrical surface model, and the computational error caused by neglecting the radial dimension of the heat source under time-varying source intensity is investigated. For the one-phase Stefan problem, both the coefficient in the solution and the development of the temperature field are presented; the computational results can be used to verify the accuracy of numerical solutions for Stefan problems.

1. Introduction

Analytical solutions for heat conduction problems have many applications. First, most of the analytical solutions are also exact solutions, thus they can be used to verify the accuracy of the numerical solutions or other approximate solutions [1–3]. Secondly, analytical solutions present explicit formulas for the temporal and spatial development of the temperature field; these explicit formulas are more advantageous to the theoretical analysis of the physical process, and they may also be used as building blocks for some numerical methods such as the unsteady surface element method [4]. Thirdly, analytical solutions are more attractive than the numerical solutions in the realm of the inverse problem analysis [5], since the solution process of the inverse problem is more convenient using the analytical solutions. Traditional methods for measuring the material thermal conductivity such as the hot wire method and the hot disk method are all designed on the basis of the inverse problem analysis using corresponding analytical solutions [6].

There are several mathematical methods that can be used to establish analytical solutions for heat conduction problems. The separation of variables technique (SVT) is a widely-used method. The SVT presents a series form of general solution for the heat conduction equation, and each term in the series is a product of several univariate functions. Basic solutions obtained from the SVT can be found in the monographs [6,7]. Beck et al. [8] indicated the issue concerning the convergence of the series and made some improvements. For complex situations such as multi-dimensional multilayer bodies, the SVT is also applicable [9]. Green's function method is another one of these mathematical methods; the method treats the internal heat source, the initial and boundary conditions as heat sources depending on time and space. The solution for the problem is obtained by integrating the Green's function with respect to time and space, and the Green's function is the temperature response induced by a unitary instantaneous point source in the corresponding geometry. The application of the Green's function method is systematically summarized in the monograph [4]. Integral transforms such as the Fourier transform, the Laplace transform can also be used to develop analytical solutions for heat conduction problems; however, the inverse transformation has to be conducted numerically in most situations, and it is a shortage of the integral transformation method [6,7].

* Corresponding author at: State Key Laboratory for Geomechanics and Deep Underground Engineering, China University of Mining & Technology, Xuzhou, Jiangsu 221116, China.

E-mail address: tod2006@126.com (Y. Zhou).

https://doi.org/10.1016/j.ijheatmasstransfer.2017.11.131

The similarity transformation method (STM) is developed based on the group theory, and it is essentially an application of the Lie groups to the heat conduction equation. The STM transforms the partial differential equation (PDE) of the heat conduction problem to an ordinary differential equation (ODE), and the solution for the heat conduction equation can be obtained from the solution of the ODE. For the heat conduction problem in a semi-infinite region under constant surface temperature or surface flux, the analytical solutions obtained from the STM can be found in the monographs [6,7]. Zahin [10] studied the heat conduction problem in a semi-infinite region under power-type initial and boundary conditions, and developed an analytical solution using the STM. For one-dimensional heat conduction process in the cylindrical coordinate, the analytical solution for Kelvin's line source model is established by the STM [6,7]. On the other hand, the STM is also widely used in the realm of the Stefan problem [11–13], which originates from the heat conduction process with phase change. The applications of the STM in recently proposed Stefan problems are summarized in Section 4.

Although many analytical solutions for heat conduction problems have been obtained using the STM, the application of the STM in the cylindrical coordinate has not been thoroughly investigated. In this paper, the main objective is to develop a similarity type of general solution for the one-dimensional heat conduction equation in the cylindrical coordinate system. After that, the general solution is applied to establish analytical solutions for two practical heat conduction problems. One problem is the infinite line source problem with power-type initial and boundary conditions, which is usually encountered during the application of the ground-coupled heat pump system. The other problem is the one-phase Stefan problem with power-type latent heat in the cylindrical coordinate, which describes a special case in the application of the artificial ground freezing technique. Computational examples for these two applications are also presented and discussed.

2. Similarity type of general solution

2.1. Solution procedure

The governing equation for the one-dimensional heat conduction process in the cylindrical coordinate can be written as

$$\frac{\partial T}{\partial t}=\alpha\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right),\quad r>0 \tag{1}$$

in which T is the temperature, r is the cylindrical coordinate, t is the time, and α is the thermal diffusivity of the material.

Eq. (1) is invariant under following group of similarity transformations [14]

$$\tilde{t}=\varepsilon^2 t,\quad \tilde{r}=\varepsilon r,\quad \tilde{T}=\varepsilon^a T \tag{2}$$

in which ε, a are group parameters (a is non-negative).

From Eq. (2), we can deduce two invariants of the group

$$\eta=\frac{T(r,t)}{2(\alpha t)^{a/2}},\quad \xi=\frac{r}{2\sqrt{\alpha t}} \tag{3}$$

Using Eq. (3), the PDE presented in Eq. (1) can be transformed to the following ODE

$$\frac{d^2\eta}{d\xi^2}+\left(2\xi+\frac{1}{\xi}\right)\frac{d\eta}{d\xi}-2a\eta=0 \tag{4}$$

We introduce a new variable z defined by

$$z=-\xi^2 \tag{5}$$

Using the variable z, Eq. (4) can be further transformed to

$$z\frac{d^2\eta}{dz^2}+(1-z)\frac{d\eta}{dz}+\frac{a}{2}\eta=0 \tag{6}$$

The detailed deduction process for Eqs. (4), (6) is shown in Appendix A.

Eq. (6) is in the form of Kummer's equation, and the general solution for this equation can be written as [15]

$$\eta(z)=AU\left(-\frac{a}{2},1,z\right)+Be^zU\left(1+\frac{a}{2},1,e^{-\pi i}z\right),\quad -\frac{1}{2}\pi\leqslant \mathrm{ph}(z)\leqslant\frac{3}{2}\pi \tag{7}$$

in which A, B are coefficients, $i=\sqrt{-1}$, and ph(z) is the argument of z.

The definition of the Kummer function $U(p, q, z)$ introduced in [16] is not applicable for Eq. (7), since $q = 1$ will lead to the evaluation of the gamma function at undefined values such as 0. The definition of $U(p, q, z)$ for the situation $q = 1, 2,\ldots$ should be written as [15]

$$\begin{aligned}U(p,q,z)=&\frac{(-1)^q}{(q-1)!\Gamma(p-q+1)}\sum_{m=0}^{\infty}\frac{(p)_m}{(q)_m m!}z^m[\ln z+\psi(p+m)-\psi(1\\&+m)-\psi(q+m)]\\&+\frac{1}{\Gamma(p)}\sum_{m=1}^{q-1}\frac{(m-1)!(1-p+m)_{q-1-m}}{(q-1-m)!}z^{-m},\quad \text{if } p\neq 0,-1,-2,\ldots\end{aligned} \tag{8}$$

$$U(p,q,z)=(-1)^p\sum_{m=0}^{-p}\binom{-p}{m}(q+m)_{-p-m}(-z)^m,\quad \text{if } p=0,-1,-2,\ldots \tag{9}$$

in which $\Gamma(\cdot)$ is the gamma function, $\psi(\cdot)$ is the digamma function, the Pochhammer symbol and the binomial coefficient are defined by

$$(p)_l=\frac{\Gamma(p+l)}{\Gamma(p)}=p(p+1)\ldots(p+l-1),\ (p)_0=1 \tag{10}$$

$$\binom{p}{m}=\frac{p!}{(p-m)!m!} \tag{11}$$

From Eqs. (3), (5), (7), the similarity type of general solution for Eq. (1) can be deduced

$$T(r,t)=2(\alpha t)^{a/2}\left[AU\left(-\frac{a}{2},1,-\xi^2\right)+Be^{-\xi^2}U\left(1+\frac{a}{2},1,-e^{-\pi i}\xi^2\right)\right],\quad -\frac{1}{2}\pi\leqslant \mathrm{ph}(-\xi^2)\leqslant\frac{3}{2}\pi \tag{12}$$

2.2. Some properties of the solution

When neither p nor $p-q+1$ is a non-positive integer, $U(p,q,z)$ can be related to the generalized hypergeometric function [15]

$$U(p,q,z)=z^{-p}{}_2F_0(p,p-q+1;-;-z^{-1}) \tag{13}$$

For the generalized hypergeometric function, there is

$$\lim_{z\to\infty}{}_2F_0(p,p-q+1;-;-z^{-1})=1 \tag{14}$$

From Eqs. (13), (14), the following equation can be deduced (neither p nor $p-q+1$ is a non-positive integer)

$$\lim_{z\to\infty}z^pU(p,q,z)=1 \tag{15}$$

For the Kummer function $U(p, q, z)$ used in Eq. (12), $p-q+1$ equals p since $q = 1$. If $p = p-q+1$ is a non-positive integer, then it can be deduced from the definition of $U(p, q, z)$ presented in Eq. (9) that Eq. (15) is still correct.