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Time two-mesh algorithm combined with finite element method for time fractional water wave model

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ABSTRACT

In this article, a new time two-mesh (TT-M) finite element (FE) method, which is constructed by a new TT-M algorithm and FE method in space, is proposed and analyzed. The numerical theories and algorithm are shown by solving the fractional water wave model including fractional derivative in time. The TT-M FE algorithm mainly covers three steps: firstly, a nonlinear FE system at some time points based on the time coarse mesh Δt_C is solved by an iterative method; further, based on the obtained numerical solution on time coarse mesh Δt_C in the first step, some useful numerical solutions between two time coarse mesh points are arrived at by the Lagrange's interpolation formula; finally, the solutions on the first and second steps are chosen as the initial iteration value, then a linear FE system on time fine mesh $\Delta t_F < \Delta t_C$ is solved. Some stable results and a priori error estimates are analyzed in detail. Furthermore, some numerical results are provided to verify the effectiveness of TT-M FE method. By the comparison with the standard FE method, it is easy to see that the CPU time can be saved by our TT-M FE method.

1. Introduction

Fractional partial differential equations (FPDEs) have been found in the fields of engineering and science. In view of the difficulty for solving the analytic solutions, increasing scholars have started to study some effective numerical methods for solving complex FPDEs with time, space and space-time fractional derivatives, which include finite difference methods [1–14], FE methods [15–26], DG methods [27–30], spectral methods [31–35], finite volume (element) methods [36–38], meshless methods [39], wavelets method [40], collocation method [41], reproducing kernel algorithm [42] and so forth.

In these mentioned numerical methods, FE methods play a significant role in finding the numerical solutions for three classes of FPDEs. So far, many researchers have given FE studies for FPDEs such as space FPDEs (Ma et al. [17] for space fractional differential equations; Zhang et al. [18] for symmetric space-fractional partial differential equations; Roop [23] for space fractional advection dispersion problem; Bu et al. [25] for Riesz space fractional diffusion equations; Zhao et al. [43] for space-fractional advection-dispersion equations; Zheng et al. [44] for space-fractional advection diffusion equation; Zhu et al. [45] for the Riesz space-fractional Fisher's equation), time FPDEs (Li et al. [16] for maxwell's equations with time fractional derivative; Jin et al. [19] for fractional order parabolic equations; Ford et al. [22] for time FPDEs; Liu et al. [24] for a time-fractional fourth-order problem; Zhuang et al. [46] for the fractional cable equation; Liu et al. [47] for a nonlinear time-fractional reaction-diffusion problem with fourth-order derivative) and space-time FPDEs (Li et al. [15] for nonlinear subdiffusion and superdiffusion equation with space-time fractional derivatives; Liu et al. [48] for space-time fractional diffusion equation; Deng [49] for the space and time fractional Fokker-Planck equation; Li and Huang [50] for the time-space fractional diffusion-wave equation). In addition to the researches on FE method for FPDEs, recently, finite element methods with two-grid algorithms [51,52] are developed for a nonlinear reaction-diffusion problem with time-fractional derivative and fourth-order derivative [53] and a nonlinear Cable equation with time-fractional derivative [54]. As there are a large number of literatures on FE methods' applications in solving FPDEs, we cannot list them all.

In this article, motivated by spatial two-grid method presented by Xu [51,52], we propose a new time two-mesh (TT-M) FE algorithm holding the advantage of saving CPU time, which includes three main computing steps: firstly, we construct a nonlinear FE system at some time points based on the time coarse mesh Δt_C, then solve this nonlinear system by an iterative method; further,

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we use interpolation formula to give useful points between any two points obtained time coarse mesh solution on the first step; finally, based on the initial iterative value computed time coarse mesh solutions, we establish a linear iterative scheme with time fine mesh Δt_F, then obtain TT-M solutions.

Here, we apply the new TT-M FE algorithm to solving the nonlinear time fractional water wave model [31,56]

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}-\beta\frac{\partial^3 u}{\partial x^2\partial t}+\frac{\nu^{\frac{1}{2}}}{\Gamma(1/2)}\int_0^t\frac{\partial u(x,\tau)}{\partial\tau}\frac{d\tau}{(t-\tau)^{\frac{1}{2}}}+\gamma\frac{\partial f(u)}{\partial x}-\alpha\frac{\partial^2 u}{\partial x^2}=0,\quad (x,t)\in\Omega\times J, \tag{1.1}$$

with boundary condition

$$u(x_L,t)=u(x_R,t)=0,\quad t\in\bar{J}, \tag{1.2}$$

and initial condition

$$u(x,0)=u_0(x),\quad x\in\Omega, \tag{1.3}$$

where $f(u)=u^2/2, J=(0,T]$ is the time interval with the positive constant $T, \Omega=[x_L,x_R](\subset R)$ is the spatial domain. $u_0(x)$ is the initial function, the coefficients $\alpha>0,\beta>0,\gamma>0,\nu\geqslant 0$ are given constants. In particular, when the coefficient ν is taken as 0, the time fractional water wave model (1.1) can be transformed into important RLW-Burgers equation. As said by Kakutani and Matsuuchi in [55], the fractional water wave model includes nonlocal pseudo-differential operators and reflects diffusive and dispersive effect stemming from the viscous layer in the fluid. In view of the importance of fractional water wave model, some researchers have done some studies with numerical algorithms. Zhang and Xu [31] gave numerical solution and theories for a water wave model with a nonlocal viscous term by using the spectral methods. Wang et al. [56] looked for the numerical solutions for fractional water wave model by combining finite difference method in time with H^1-Galerkin MFE procedure in space.

In this article, we will discuss the detailed numerical theories of new TT-M FE method by solving numerically nonlinear fractional water wave model. We will give the detailed analysis on the stability and error estimates in L^2-norm, then provide some numerical calculations to test and verify the effectiveness and feasibility for our TT-M FE method. From our calculating results with comparison to standard nonlinear FE method, ones can see that the TT-M FE method not only maintains the computational accuracy, but also save the CPU time.

The structure of the paper is as follows. In Section 2, we give the numerical scheme for new TT-M FE method. In Section 3, we prove the stability on time coarse mesh and TT-M FE method. In Section 4, we derive the error estimates on both time coarse mesh method with Δt_C and TT-M FE method. In Section 5, we implement the numerical calculation by using TT-M FE method and standard nonlinear FE method, and make some comparisons between two numerical methods. Finally, in Section 6, we make some conclusions and future advancements for our methods. In the full text, we use some constants $\mathbb{C}$, which are free of space mesh h, time coarse mesh Δt_C, time fine mesh Δt_F and may be different in different places.

2. Numerical scheme

In order to arrive at the fully discrete TT-M FE scheme, we need to split the time interval $[0,T]$ into uniform partition with the nodes $t_n=n\Delta t(n=0,1,2,\cdots,\mathcal{N})$, which satisfies $0=t_0<t_1<t_2<\cdots<t_N=T$ with mesh length $\Delta t=T/\mathcal{N}$ for some positive integer $\mathcal{N}$. Now we define $\phi^n=\phi(t_n)$ for a smooth function ϕ on $[0,T]$ and the notation $\partial_{\Delta t}[\phi^{n+1}]\triangleq\frac{3\phi^{n+1}-4\phi^n+\phi^{n-1}}{2\Delta t}$ used in [47].

For formulating the discrete scheme, we give the following equality for the 1/2-order fractional derivative at time $t=t_{n+1}$

$$\frac{1}{\Gamma(1/2)}\int_0^{t_{n+1}}\frac{\partial u(x,\tau)}{\partial\tau}\frac{d\tau}{(t_{n+1}-\tau)^{\frac{1}{2}}}=\frac{1}{\Gamma(3/2)}\sum_{k=0}^{n}\mathcal{Q}_k^{1/2}\frac{u(t_{n+1-k})-u(t_{n-k})}{\Delta t^{1/2}}+\varepsilon_0^{n+1}, \tag{2.1}$$

where $\mathcal{Q}_k^{1/2}=(k+1)^{1/2}-(k)^{1/2}$ and ε_0^{n+1} is the truncation error including the following estimate with the positive constant $\mathbb{C}$ (see [32,47,57])

$$\|\varepsilon_0^{n+1}\|\leqslant\mathbb{C}\Delta t^{3/2}. \tag{2.2}$$

Using second-order backward difference approximation, then applying Green's formula, we find $u^{(m+1)s}:[0,T]\mapsto H_0^1=\{v|v\in H^1(\Omega), v(x_L)=v(x_R)=0\}$ to obtain the weak formulation of (1.1)–(1.3) for any $v\in H_0^1$ as

Case $m=0$:

$$\left(\frac{u^s-u^0}{\Delta t_{\mathcal{G}}},v\right)+\beta\left(\frac{u_x^s-u_x^0}{\Delta t_{\mathcal{G}}},v_x\right)+\delta(\nu)\mathcal{Q}_0^{1/2}\left(\frac{u^s-u^0}{\Delta t_{\mathcal{G}}^{1/2}},v\right)+\alpha(u_x^s,v_x)-(u^s,v_x)=\gamma(f(u^s),v_x)+(\varepsilon_0^s,v)+(\varepsilon_1^s,v)+(\varepsilon_2^s,v_x), \tag{2.3}$$

where u^s is the first weak solution at time t_s calculated by the initial value $u^0, \Delta t_{\mathcal{G}}=t_s-t_0=t_{(m+1)s}-t_{ms}(\mathcal{G}=F \text{ or } C)$ is the time step length, and

$$\begin{aligned}\varepsilon_1^s&=\tfrac{u^s-u^0}{\Delta t_{\mathcal{G}}}-\tfrac{\partial u}{\partial t}(t_s)+O(\Delta t_{\mathcal{G}}),\\ \varepsilon_2^s&=\tfrac{u_x^s-u_x^0}{\Delta t_{\mathcal{G}}}-\tfrac{\partial^2 u}{\partial x\partial t}(t_s)+O(\Delta t_{\mathcal{G}}).\end{aligned} \tag{2.4}$$

Case $m\geqslant 1$:

$$\left(\partial_{\Delta t_{\mathcal{G}}}[u^{(m+1)s}],v\right)+\beta\left(\partial_{\Delta t_{\mathcal{G}}}[u_x^{(m+1)s}],v_x\right)+\delta(\nu)\sum_{k=0}^{m}\mathcal{Q}_k^{1/2}\times\left(\frac{u(t_{(m+1-k)s})-u(t_{(m-k)s})}{\Delta t_{\mathcal{G}}^{1/2}},v\right)+\alpha(u_x^{(m+1)s},v_x)-(u^{(m+1)s},v_x)=\gamma(f(u^{(m+1)s}),v_x)+(\varepsilon_0^{(m+1)s},v)+(\varepsilon_1^{(m+1)s},v)+(\varepsilon_2^{(m+1)s},v_x), \tag{2.5}$$

where $\delta(\nu)=\frac{\nu^{\frac{1}{2}}}{\Gamma(3/2)}$ and

$$\begin{aligned}\varepsilon_1^{(m+1)s}&=\partial_{\Delta t_{\mathcal{G}}}[u^{(m+1)s}]-\tfrac{\partial u}{\partial t}(t_{(m+1)s})+O(\Delta t_{\mathcal{G}}^2),\\ \varepsilon_2^{(m+1)s}&=\partial_{\Delta t_{\mathcal{G}}}[u_x^{(m+1)s}]-\tfrac{\partial^2 u}{\partial x\partial t}(t_{(m+1)s})+O(\Delta t_{\mathcal{G}}^2).\end{aligned} \tag{2.6}$$

For formulating FE scheme, we define V_h as the FE subspace of H_0^1. Then, we find $u_h^{(m+1)s}\in V_h$ to formulate a standard nonlinear FE system for any $v_h\in V_h$ as

Case $m=0$:

$$\left(\frac{u_h^s-u_h^0}{\Delta t_{\mathcal{G}}},v_h\right)+\beta\left(\frac{u_{hx}^s-u_{hx}^0}{\Delta t_{\mathcal{G}}},v_{hx}\right)+\delta(\nu)\mathcal{Q}_0^{1/2}\left(\frac{u_h^s-u_h^0}{\Delta t_{\mathcal{G}}^{1/2}},v_h\right)+\alpha(u_{hx}^s,v_{hx})-(u_h^s,v_{hx})=\gamma(f(u_h^s),v_{hx}), \tag{2.7}$$

where u_h^s is the first numerical solution at time t_s computed by the initial value u_h^0.

Case $m\geqslant 1$:

$$\left(\partial_{\Delta t_{\mathcal{G}}}[u_h^{(m+1)s}],v_h\right)+\beta\left(\partial_{\Delta t_{\mathcal{G}}}[u_{hx}^{(m+1)s}],v_{hx}\right)+\delta(\nu)\sum_{k=0}^{m}\mathcal{Q}_k^{1/2}\left(\frac{u_h(t_{(m+1-k)s})-u_h(t_{(m-k)s})}{\Delta t_{\mathcal{G}}^{1/2}},v_h\right)+\alpha(u_{hx}^{(m+1)s},v_{hx})-(u_h^{(m+1)s},v_{hx})=\gamma(f(u_h^{(m+1)s}),v_{hx}). \tag{2.8}$$